On the approximation of general shell problems by the clough-johnson flat plate elements. Part 2 : study of the convergence and error estimates

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Rapports de Recherche

N° 667

ON THE APPROXIMATION OF GENERAL SHELL PROBLEMS BY THE CLOUGH-JOHNSON FLAT PLATE ELEMENTS

PART 2: STUDY OF THE CONVERGENCE AND ERROR ESTIMATES

Michel BERNADOU
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Mai 1987

ON THE APPROXIMATION OF GENERAL SHELL PROBLEMS BY THE CLOUGH-JOHNSON FLAT PLATE ELEMENTS

PART 2 : STUDY OF THE CONVERGENCE AND ERROR ESTIMATES

Michel BERNADOU[(1)] - Yves DUCATEL[(2)] - Pascal TROUVE[(1)]

Summary : In this series of reports, we give an account of some results obtained in the approximation of *general shell problems* by the CLOUGH-JOHNSON *flat plate elements*. The first part is concerned by the study of the *compatibility equations*. In the second part, we deliver several interesting results valid for general shells and we prove the "pseudo-convergence" of the method for a class of shallow shells. Then, this careful study allows us to introduce a perturbation of this approximation and thus, to propose a new method which is convergent for general shells. Finally, in the third part, we describe in details how to implement the CLOUGH-JOHNSON method.

SUR L'APPROXIMATION DE PROBLEMES GENERAUX DE COQUES PAR DES METHODES D'ELEMENTS FINIS PLATS DE CLOUGH ET JOHNSON

PARTIE 2 : ETUDE DE LA CONVERGENCE ET ESTIMATIONS D'ERREUR

Résumé : Dans cette série de rapports, nous rassemblons les divers résultats obtenus dans l'approximation de *problèmes généraux de coques* par les éléments finis plats (de plaques) de CLOUGH et JOHNSON. La première partie est relative à l'étude des équations de *compatibilité*. Dans la seconde partie, nous donnons plusieurs résultats intéressants valables pour des coques générales, puis nous démontrons la pseudo-convergence de la méthode pour une classe de coques peu profondes. Cette étude détaillée nous permet alors d'introduire une perturbation de cette approximation et ainsi, de proposer une nouvelle méthode qui converge pour des coques générales. Finalement, dans la troisième partie, nous détaillons la marche à suivre pour implémenter la méthode de CLOUGH et JOHNSON.

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CONTENTS

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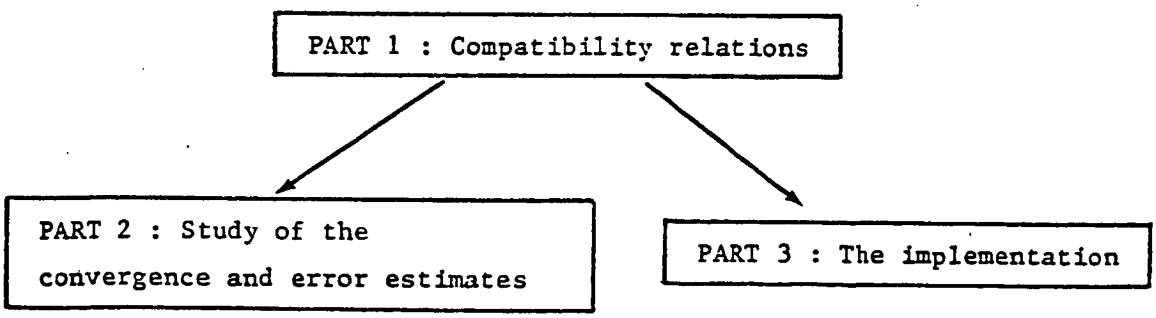
HOW TO READ THE PAPER

All the paper is divided in three parts. In the first one the reader can find all the basis. Then, *part 2 and part 3 can be read independtly* depending of the interest of everyone. In other words we can summarize these possibilities in the following "flow shart" :



ACKNOWLEDGEMENT : The Authors are indebted to F. PALMA who read the manuscript in its entirely and who suggest various improvements ; also, they gratefully appreciate the excellent typing of Mrs. DESNOUS.

5 - THE CONSISTENCY ERRORS

In this part II, we shall be interested in the following problems :

(i) show that the problem (4.3.19) has a unique solution $\vec{u}_h^* \in \vec{V}_h$: this will be achieved by showing that, under mild assumptions, the bilinear form $b_h(.,.)$ is $\vec{V}_h$-elliptic, uniformly with respect to h : that will be the purpose of Theorem 5.4.1;

(ii) find sufficient conditions on numerical integration schemes which ensure that (the norm on $\vec{V}$ is defined by (2.2.2)) :

$$||\vec{u} - \vec{u}_h^*||_{\vec{V}} = ||\vec{u} - \vec{u}_h^*|| = O(h) \ , \qquad (5.0.1)$$

i.e., the same order as in (3.2.2). In this estimate, $\vec{u}$ (resp. $\vec{u}_h^*$) denotes the solution of the continuous problem (2.2.5) (resp. of the approximate problem (4.3.19)).

5.1 - Abstract error estimate

In order to solve the problem (5.0.1), we first give an "abstract" error estimate.

Theorem 5.1.1 : Let us consider a family of discrete problems (4.3.19) for which the bilinear forms are $\vec{V}_h$-elliptic, uniformly with respect to h, in the sense that there exists a constant $\alpha > 0$, independent of h, such that

$$\alpha \ ||\vec{v}_h||^2 \leq b_h(\vec{v}_h,\vec{v}_h) \quad , \quad \forall \ \vec{v}_h \in \vec{V}_h \ . \qquad (5.1.1)$$

Then, there exists a constant c, independent of h, such that

$$\left.\begin{aligned} \|\vec{u}-\vec{u}_h^*\| \le c\ \{ \inf_{\vec{v}_h\in\vec{V}_h} \{ \|\vec{u}-\vec{v}_h\| &+ \sup_{\vec{w}_h\in\vec{V}_h} \frac{|a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|} \} \\ &+ \sup_{\vec{w}_h\in\vec{V}_h} \frac{|f(\vec{w}_h)-g_h(\vec{w}_h)|}{\|\vec{w}_h\|} \} \end{aligned}\right\} \quad (5.1.2)$$

<u>where</u> $\vec{u}$ <u>(resp.</u> $\vec{u}_h^*$<u>) denotes the solution of the continuous problem (2.2.5)</u> <u>(resp. of the discrete problem (4.3.19))</u>.

<u>Proof</u> : The assumption of $\vec{V}_h$-ellipticity involves the existence and the uniqueness of a solution $\vec{u}_h^*$ for the discrete problem (4.3.19). Then, let $\vec{v}_h$ be any element of the space $\vec{V}_h$. We can write

$$\begin{aligned} \alpha\ \|\vec{u}_h^*-\vec{v}_h\|^2 &\le b_h(\vec{u}_h^*-\vec{v}_h,\vec{u}_h^*-\vec{v}_h) \\ &= a(\vec{u}-\vec{v}_h,\vec{u}_h^*-\vec{v}_h) + [a(\vec{v}_h,\vec{u}_h^*-\vec{v}_h)-b_h(\vec{v}_h,\vec{u}_h^*-\vec{v}_h)] \\ &\quad + [g_h(\vec{u}_h^*-\vec{v}_h) - f(\vec{u}_h^*-\vec{v}_h)]\ , \end{aligned}$$

so that the continuity of the bilinear form $a(.,.)$ involves

$$\begin{aligned} \alpha\ \|\vec{u}_h^*-\vec{v}_h\| &\le M\|\vec{u}-\vec{v}_h\| + \frac{|a(\vec{v}_h,\vec{u}_h^*-\vec{v}_h)-b_h(\vec{v}_h,\vec{u}_h^*-\vec{v}_h)|}{\|\vec{u}_h^*-\vec{v}_h\|} \\ &\quad + \frac{|f(\vec{u}_h^*-\vec{v}_h) - g_h(\vec{u}_h^*-\vec{v}_h)|}{\|\vec{u}_h^*-\vec{v}_h\|} \\ &\le M\|\vec{u}-\vec{v}_h\| + \sup_{\vec{w}_h\in\vec{V}_h} \frac{|a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h)|}{\|\vec{w}_h\|} \\ &\quad + \sup_{\vec{w}_h\in\vec{V}_h} \frac{|f(\vec{w}_h)-g_h(\vec{w}_h)|}{\|\vec{w}_h\|} \end{aligned}$$

Combining this inequality with the triangular inequality

$$||\vec{u}-\vec{u}_h^*|| \leq ||\vec{u}-\vec{v}_h|| + ||\vec{v}_h-\vec{u}_h^*|| \quad ,$$

and taking the minimum with respect to $\vec{v}_h \in \vec{V}_h$, we get the inequality (5.1.2). □

Let us remark that, in estimate (5.1.2), we find in addition to the usual approximation theory term $\inf\limits_{\vec{v}_h\in\vec{\vec{V}}_h} ||\vec{u}-\vec{v}_h||$, two additional terms which measure the *consistency* between the bilinear forms $a(.,.)$ and $b_h(.,.)$, on the one hand, and between the linear forms $f(.)$ and $g_h(.)$, on the other hand.

5.2 - Estimate of the consistency error $|a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h)|$

Let us consider any functions $\vec{v}_h,\vec{w}_h \in \vec{X}_h$ and let us denote $\vec{\vec{v}}_h$, $\vec{\vec{w}}_h \in \vec{\vec{X}}_h$ the corresponding functions through the bijection F_h defined in theorem 4.2.1. Using the definitions (4.3.3) (4.3.14) (4.3.16) we have

$$a(\vec{v}_h,\vec{w}_h)-b_h(\vec{v}_h,\vec{w}_h) = a(\vec{v}_h,\vec{w}_h) - \tilde{a}_h(\vec{\vec{v}}_h,\vec{\vec{w}}_h) + \tilde{a}_h(\vec{\vec{v}}_h,\vec{\vec{w}}_h)-a_h^*(\vec{\vec{v}}_h,\vec{\vec{w}}_h). \quad (5.2.1)$$

To evaluate the consistency error we have firstly to consider the following estimate :

Estimate of $|a(\vec{v}_h,\vec{w}_h)-\tilde{a}_h(\vec{\vec{v}}_h,\vec{\vec{w}}_h)|$: From (2.2.3) and (4.3.3) we record that for any $\vec{v}_h$, $\vec{w}_h \in \vec{X}_h$ and for any $\vec{\vec{v}}_h,\vec{\vec{w}}_h \in \vec{\vec{X}}_h$ we have

$$\left.\begin{aligned} a(\vec{v}_h,\vec{w}_h) = \sum_{K\in\mathcal{C}_h} \int_K \frac{Ee}{1-\nu^2} \{ &(1-\nu)\gamma_\beta^\alpha(\vec{v}_h)\gamma_\alpha^\beta(\vec{w}_h) + \nu\, \gamma_\alpha^\alpha(\vec{v}_h)\gamma_\beta^\beta(\vec{w}_h) \\ &+ \frac{e^2}{12} [(1-\nu)\bar{\rho}_\beta^\alpha(\vec{v}_h)\bar{\rho}_\alpha^\beta(\vec{w}_h) + \nu\, \bar{\rho}_\alpha^\alpha(\vec{v}_h)\bar{\rho}_\beta^\beta(\vec{w}_h)]\} \sqrt{a}\, d\xi^1\, d\xi^2 \end{aligned}\right\} (5.2.2)$$

and

$$\left.\begin{aligned}\tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h) = \sum_{K\in\mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2}\{(1-\nu)\tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h)\tilde{\gamma}^{\beta}_{h\alpha}(\tilde{\vec{w}}_h) + \nu\,\tilde{\gamma}^{\alpha}_{h\alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}^{\beta}_{h\beta}(\tilde{\vec{w}}_h)\\ + \frac{e^2}{12}[(1-\nu)\tilde{\rho}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h)\tilde{\rho}^{\beta}_{h\alpha}(\tilde{\vec{w}}_h) + \nu\,\tilde{\rho}^{\alpha}_{h\alpha}(\tilde{\vec{v}}_h)\tilde{\rho}^{\beta}_{h\beta}(\tilde{\vec{w}}_h)]\}\sqrt{a_h}\,d\xi^1 d\xi^2\end{aligned}\right\} \quad (5.2.3)$$

so that

$$|a(\vec{v}_h,\vec{w}_h)-\tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \le \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{4}|ER_{Ki}(\vec{v}_h,\vec{w}_h)| \quad (5.2.4)$$

with

$$\left.\begin{aligned}ER_{K1}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee}{1+\nu}[\gamma^{\alpha}_{\beta}(\vec{v}_h)\gamma^{\beta}_{\alpha}(\vec{w}_h)\sqrt{a}\\ - \tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h)\tilde{\gamma}^{\beta}_{h\alpha}(\tilde{\vec{w}}_h)\sqrt{a_h}]\,d\xi^1 d\xi^2\end{aligned}\right\} \quad (5.2.5)$$

$$\left.\begin{aligned}ER_{K2}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee\nu}{1-\nu^2}[\gamma^{\alpha}_{\alpha}(\vec{v}_h)\gamma^{\beta}_{\beta}(\vec{w}_h)\sqrt{a}\\ - \tilde{\gamma}^{\alpha}_{h\alpha}(\tilde{\vec{v}}_h)\tilde{\gamma}^{\beta}_{h\beta}(\tilde{\vec{w}}_h)\sqrt{a_h}]\,d\xi^1 d\xi^2\end{aligned}\right\} \quad (5.2.6)$$

$$\left.\begin{aligned}ER_{K3}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee^3}{12(1+\nu)}[\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)\bar{\rho}^{\beta}_{\alpha}(\vec{w}_h)\sqrt{a}\\ - \tilde{\rho}^{\alpha}_{h\beta}(\vec{v}_h)\tilde{\rho}^{\beta}_{h\alpha}(\vec{w}_h)\sqrt{a_h}]\,d\xi^1 d\xi^2\end{aligned}\right\} \quad (5.2.7)$$

$$\left.\begin{aligned}ER_{K4}(\vec{v}_h,\vec{w}_h) = \int_K \frac{Ee^3\nu}{12(1-\nu^2)}[\bar{\rho}^{\alpha}_{h\alpha}(\vec{v}_h)\bar{\rho}^{\beta}_{h\beta}(\vec{w}_h)\sqrt{a}\\ - \tilde{\rho}^{\alpha}_{h\alpha}(\tilde{\vec{v}}_h)\tilde{\rho}^{\beta}_{h\beta}(\tilde{\vec{w}}_h)\sqrt{a_h}]\,d\xi^1 d\xi^2\end{aligned}\right\} \quad (5.2.8)$$

If we consider the term $ER_{K1}(\vec{v}_h,\vec{w}_h)$, we obtain

$$
\left.
\begin{aligned}
&|ER_{K1}(\vec{v}_h,\vec{w}_h)| \le \\
&\quad |\tfrac{Ee}{1+\nu}|_{0,\infty,K} \{ |\sqrt{a}-\sqrt{a}_h|_{0,\infty,K} [|\gamma^\alpha_\beta(\vec{v}_h)|_{0,K} \, |\gamma^\beta_\alpha(\vec{w}_h)|_{0,K} + \\
&\quad + |\gamma^\alpha_\beta(\vec{v}_h)|_{0,K} |\gamma^\beta_\alpha(\vec{w}_h)-\tilde{\gamma}^{\ \beta}_{h\alpha}(\tilde{\vec{w}}_h)|_{0,K} + \\
&\quad + |\gamma^\alpha_\beta(\vec{w}_h)|_{0,K} |\gamma^\beta_\alpha(\vec{v}_h)-\tilde{\gamma}^{\ \beta}_{h\alpha}(\tilde{\vec{v}}_h)|_{0,K} + \\
&\quad + |\gamma^\alpha_\beta(\vec{v}_h)-\tilde{\gamma}^{\ \alpha}_{h\beta}(\tilde{\vec{v}}_h)|_{0,K} |\gamma^\beta_\alpha(\vec{w}_h)-\tilde{\gamma}^{\ \beta}_{h\alpha}(\tilde{\vec{w}}_h)|_{0,K}] \\
&\quad + |\sqrt{a}|_{0,\infty,K} [|\gamma^\alpha_\beta(\vec{v}_h)|_{0,K} |\gamma^\beta_\alpha(\vec{w}_h)-\tilde{\gamma}^{\ \beta}_{h\alpha}(\tilde{\vec{w}}_h)|_{0,K} + \\
&\quad + |\gamma^\alpha_\beta(\vec{w}_h)|_{0,K} |\gamma^\beta_\alpha(\vec{v}_h)-\tilde{\gamma}^{\ \beta}_{h\alpha}(\tilde{\vec{v}}_h)|_{0,K} \\
&\quad + |\gamma^\alpha_\beta(\vec{v}_h)-\tilde{\gamma}^{\ \alpha}_{h\beta}(\tilde{\vec{v}}_h)|_{0,K} |\gamma^\beta_\alpha(\vec{w}_h)-\tilde{\gamma}^{\ \beta}_{h\alpha}(\tilde{\vec{w}}_h)|_{0,K}]\}
\end{aligned}
\right\} \quad (5.2.9)
$$

and similar results for the others ER_{Ki}.

Then, a convenient way to estimate the expressions $ER_{Ki}(\vec{v}_h,\vec{w}_h)$ consists to evaluate the terms

$$|\gamma^\alpha_\beta(\vec{v}_h)-\tilde{\gamma}^\alpha_{h\beta}(\tilde{\vec{v}}_h)|_{0,K} \quad , \quad |\bar{\rho}^\alpha_\beta(\vec{v}_h)-\tilde{\rho}^\alpha_{h\beta}(\tilde{\vec{v}}_h)|_{0,K} \quad , \quad |\sqrt{a}-\sqrt{a_h}|_{0,\infty,K} \ . \qquad (5.2.10)$$

This is the object of theorem 5.2.1 to theorem 5.2.3.

From the definitions (2.1.11), (2.1.13) and (4.3.1), we obtain

$$\gamma^\alpha_\beta(\vec{v}_h) = a^{\alpha\lambda} [\tfrac{1}{2}(v_{h\beta,\lambda} + v_{h\lambda,\beta}) - \Gamma^\nu_{\lambda\beta} v_{h\nu} - b_{\lambda\beta} v_{h3}] \qquad (5.2.11)$$

$$\tilde{\gamma}^\alpha_{h\beta}(\tilde{\vec{v}}_h) = a^{\alpha\nu}_h \, \tilde{\gamma}_{h\beta\nu}(\tilde{\vec{v}}_h) = \frac{1}{2} a^{\alpha\nu}_h (\tilde{v}_{h\beta,\nu} + \tilde{v}_{h\nu,\beta}) \ . \qquad (5.2.12)$$

Now let us prove the following theorem :

Theorem 5.2.1 : There exists a constant c independent of h such that for any $\tilde{\vec{v}}_h \in \tilde{\vec{X}}_h$ and $\vec{v}_h \in \vec{X}_h$ in correspondence through the bijection F_h defined in Theorem 4.2.1, we have

$$|\gamma^\alpha_\beta(\vec{v}_h) - \tilde{\gamma}^\alpha_{h\beta}(\tilde{\vec{v}}_h)|_{0,K} \le c\, h \, \{||v_{h1}||^2_{1,K} + ||v_{h2}||^2_{1,K} + ||v_{h3}||^2_{1,K}\}^{1/2} \ . \qquad (5.2.13)$$

Proof (in four steps) :

Step 1 : An expression of $\tilde{v}_{h\beta,\lambda}$ as function of degrees of freedom of space $\vec{\tilde{X}}_h$

For any function $\vec{\tilde{v}}_h$ of $\vec{\tilde{X}}_h$, let us consider its restriction $\vec{\tilde{v}}_h|_K$ to a triangle K of $\mathcal{T}_h$, still denoted $\vec{\tilde{v}}_h$ for simplicity. By definition $\tilde{v}_{h\beta} \in P_1(K)$ so that, for any $\xi = (\xi^1,\xi^2) \in K$:

$$\tilde{v}_{h\beta}(\xi) = \sum_{i=1}^{3} \lambda_i \, \tilde{v}_{h\beta}(\Sigma_i) \tag{5.2.14}$$

and

$$\tilde{v}_{h\beta,\nu}(\xi) = \sum_{i=1}^{3} \tilde{v}_{h\beta}(\Sigma_i) \frac{\partial \lambda_i}{\partial \xi^\nu} \tag{5.2.15}$$

where we denote $\Sigma_i = (\xi_i^1,\xi_i^2)$, $1 \le i \le 3$, the vertices of the considered triangle K.

By virtue of the compatibility relations (4.2.13), we have $\vec{\tilde{v}}_h(\Sigma_i) = \vec{v}_h(\Sigma_i)$. Since by definition $\vec{\tilde{v}}_h(\xi) = \tilde{v}_{hj}(\xi)\vec{a}_h^j$ and $\vec{v}_h(\xi) = v_{hj}(\xi)\vec{a}^j(\xi)$, we obtain

$$\tilde{v}_{h\beta}(\Sigma_i) = d_{h\beta}^j(\Sigma_i) v_{hj}(\Sigma_i) \quad , \quad i=1,2,3 , \tag{5.2.16}$$

where we have denoted

$$d_{hk}^j(\xi) = \vec{a}^j(\xi).\vec{a}_{hk} \quad , \quad \xi \in K . \tag{5.2.17}$$

Combining the relations (5.2.15), (5.2.16) and (5.2.17), we deduce

$$\tilde{v}_{h\beta,\nu}(\xi) = \sum_{i=1}^{3} d_{h\beta}^j(\Sigma_i) \frac{\partial \lambda_i}{\partial \xi^\nu} v_{hj}(\Sigma_i) . \tag{5.2.18}$$

Step 2 : Finite expansions of some terms appearing in (5.2.18)

In the following, we denote c, c_i, c^j, c_i^j, ... constants which are independent of h.

The assumption $\vec{\phi} \in (\mathscr{C}^3(\Omega))^3$ permits to write

$$\left.\begin{aligned} \vec{a}^j(\Sigma_i) &= \vec{a}^j(\xi) + (\xi_i^\varepsilon-\xi^\varepsilon)\vec{a}^j_{,\varepsilon}(\xi) + 0(h^2)\vec{c}_i^j \\ \vec{a}_j(\Sigma_i) &= \vec{a}_j(\xi) + (\xi_i^\varepsilon-\xi^\varepsilon)\vec{a}_{j,\varepsilon}(\xi) + 0(h^2)\vec{c}_{ji} \ . \end{aligned}\right\} \tag{5.2.19}$$

Using the Gauss and Weingarten relations (GREEN and ZERNA [20]), i.e.

$$\left.\begin{aligned} \vec{a}_{\alpha,\beta} &= \Gamma^\nu_{\alpha\beta}\,\vec{a}_\nu + b_{\alpha\beta}\,\vec{a}_3 \\ \vec{a}^\alpha_{,\beta} &= -\,\Gamma^\alpha_{\beta\nu}\,\vec{a}^\nu + b^\alpha_\beta\,\vec{a}^3 \end{aligned}\right\} \text{(Gauss)} \tag{5.2.20}$$

$$\vec{a}_{3,\alpha} = \vec{a}^3_{,\alpha} = -\,b^\nu_\alpha\,\vec{a}_\nu \ , \quad \text{(Weingarten)} \tag{5.2.21}$$

we finally derive

$$\left.\begin{aligned} \vec{a}^\alpha(\Sigma_i) &= \vec{a}^\alpha(\xi) - (\xi_i^\varepsilon-\xi^\varepsilon)\Gamma^\alpha_{\varepsilon\nu}\vec{a}^\nu(\xi) + (\xi_i^\varepsilon-\xi^\varepsilon)b^\alpha_\varepsilon\vec{a}_3 + 0(h^2)\vec{c}_i^\alpha \ , \\ \vec{a}_\alpha(\Sigma_i) &= \vec{a}_\alpha(\xi) + (\xi_i^\varepsilon-\xi^\varepsilon)\Gamma^\nu_{\varepsilon\alpha}\vec{a}_\nu(\xi) + (\xi_i^\varepsilon-\xi^\varepsilon)b_{\alpha\varepsilon}\vec{a}_3 + 0(h^2)\vec{c}_{\alpha i} \ , \\ \vec{a}_3(\Sigma_i) &= \vec{a}_3(\xi) - (\xi_i^\varepsilon-\xi^\varepsilon)b^\nu_\varepsilon\,\vec{a}_\nu(\xi) + 0(h^2)\vec{c}_{3i} \ . \end{aligned}\right\} \tag{5.2.22}$$

The relations between the basis $\vec{a}_i$, $\vec{a}^i$, $\vec{a}_{hi}$, $\vec{a}_h^i$ are readily obtained from the definition of the mapping $\vec{\phi}_h$: by definition, we have at any point $\xi \in K$

$$\vec{\phi}_h(\xi) = \sum_{i=1}^{3} \lambda_i\,\vec{\phi}_h(\Sigma_i) = \sum_{i=1}^{3} \lambda_i\,\vec{\phi}(\Sigma_i)$$

and thus

$$\vec{a}_{h\beta} = \vec{\phi}_{h,\beta}(\xi) = \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\beta} \vec{\phi}(\Sigma_i) \ . \tag{5.2.23}$$

But,

$$\left.\begin{aligned} \vec{\phi}(\Sigma_i) = \vec{\phi}(\xi) + (\xi_i^\varepsilon - \xi^\varepsilon) \frac{\partial\vec{\phi}}{\partial\xi^\varepsilon}(\xi) + \frac{1}{2}(\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta)\frac{\partial^2\vec{\phi}}{\partial\xi^\varepsilon\partial\xi^\eta}(\xi) \\ + 0(h^3)\vec{c}_i \quad , \quad \forall\ \xi \in K \ . \end{aligned}\right\} \tag{5.2.24}$$

Let us notice that

$$\left.\begin{aligned} &\sum_{i=1}^{3} \lambda_i = 1 \quad , \quad \sum_{i=1}^{3} \lambda_i(\xi_i^\varepsilon-\xi^\varepsilon) = 0 \quad , \quad \sum_{i=1}^{3} (\xi_i^\varepsilon-\xi^\varepsilon)\frac{\partial\lambda_i}{\partial\xi^\beta} = \delta_\beta^\varepsilon \ , \\ &\frac{\partial\lambda_i}{\partial\xi^\beta} = 0(h^{-1})c_\beta^i \ . \end{aligned}\right\} \tag{5.2.25}$$

If we set

$$A_\beta^{\varepsilon\eta}(\xi) = \frac{1}{2} \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\beta} (\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta) \ , \tag{5.2.26}$$

we obtain by combining relations (5.2.20) and (5.2.23) to (5.2.26)

$$\left.\begin{aligned} \vec{a}_{h\beta} = \vec{a}_\beta(\xi) + A_\beta^{\varepsilon\eta}(\xi) [\, \Gamma_{\varepsilon\eta}^\nu(\xi)\vec{a}_\nu(\xi) + b_{\varepsilon\eta}(\xi)\vec{a}_3(\xi)] + 0(h^2)\vec{c}_\beta \ , \\ \forall\ \xi \in K \ . \end{aligned}\right\} \tag{5.2.27}$$

Then, from (5.2.22) and (5.2.27), we derive with (5.2.17) and (5.2.26)

$$\left.\begin{aligned} &d_{h\beta}^\alpha(\Sigma_i) = \delta_\beta^\alpha - (\xi_i^\varepsilon-\xi^\varepsilon)\Gamma_{\varepsilon\beta}^\alpha(\xi) + A_\beta^{\varepsilon\eta}(\xi)\Gamma_{\varepsilon\eta}^\alpha(\xi) + 0(h^2)c_{\beta i}^\alpha \ , \\ &d_{h\beta}^3(\Sigma_i) = b_{\varepsilon\eta}(\xi)A_\beta^{\varepsilon\eta}(\xi) - (\xi_i^\varepsilon-\xi^\varepsilon)b_{\varepsilon\beta}(\xi) + 0(h^2)c_{\beta i}^3 \ , \\ &\qquad \forall\ \xi \in K \ . \end{aligned}\right\} \tag{5.2.28}$$

Finally, since $v_{h\alpha} \in P_1(K)$, $v_{h3} \in P_3(K_i)$ (K_i = subtriangle of K, $\bar{K} = \bigcup_{i=1}^{3} \bar{K}_i$) and $v_{h3} \in \mathscr{C}^1(K)$, we have for any $\xi \in K$:

$$\left.\begin{aligned} v_{h\alpha}(\Sigma_i) &= v_{h\alpha}(\xi) + (\xi_i^\varepsilon - \xi^\varepsilon) v_{h\alpha,\varepsilon} \\ v_{h3}(\Sigma_i) &= v_{h3}(\xi) + (\xi_i^\varepsilon - \xi^\varepsilon) v_{h3,\varepsilon}(\bar{\xi}_i) \ , \text{ where } \ \bar{\xi}_i \in [\Sigma_i, \xi]. \end{aligned}\right\} \tag{5.2.29}$$

<u>Step 3</u> : <u>Finite expansion of</u> $\tilde{\gamma}^\alpha_{h\beta}(\vec{v}_h)$:

Combining finite expansions (5.2.28) (5.2.29) with the expression (5.2.18) and using relations (5.2.25), we obtain :

$$\begin{aligned} \tilde{v}_{h\beta,\lambda}(\xi) &= v_{h\beta,\lambda}(\xi) - \Gamma^\nu_{\beta\lambda}(\xi)\, v_{h\nu}(\xi) - b_{\beta\lambda}(\xi) v_{h3}(\xi) \\ &\quad + 0(h)\ \{c^j_{\lambda\beta}\, v_{hj}(\xi) + c^{\nu\varepsilon}_{\lambda\beta}\, v_{h\nu,\varepsilon} + \sum_{i=1}^{3} c^\varepsilon_{\lambda\beta i}\, v_{h3,\varepsilon}(\bar{\xi}_i)\} \ . \end{aligned}$$

Substituting the previous finite expansion into the expression (5.2.12) and using (5.2.27), we finally deduce

$$\begin{aligned} \tilde{\gamma}^\alpha_{h\beta}(\vec{v}_h) &= a^{\alpha\lambda}(\xi)\ \{\tfrac{1}{2}\,[v_{h\beta,\lambda}(\xi) + v_{h\lambda,\beta}(\xi)] \\ &\qquad - \Gamma^\nu_{\lambda\beta}(\xi) v_{h\nu}(\xi) - b_{\lambda\beta}(\xi) v_{h3}(\xi)\} \\ &\quad + 0(h)\ \{c^{\alpha j}_\beta\, v_{hj}(\xi) + c^{\alpha\nu\varepsilon}_\beta\, v_{h\nu,\varepsilon} + \sum_{i=1}^{3} c^{\alpha\varepsilon}_{\beta i}\, v_{h3,\varepsilon}(\bar{\xi}_i)\} \end{aligned}$$

or, using the definition (5.2.11)

$$\tilde{\gamma}^\alpha_{h\beta}(\vec{v}_h) = \gamma^\alpha_\beta(\vec{v}_h) + 0(h)\ \{c^{\alpha j}_\beta\, v_{hj}(\xi) + c^{\alpha\nu\varepsilon}_\beta\, v_{h\nu,\varepsilon} + \sum_{i=1}^{3} c^{\alpha\varepsilon}_{\beta i}\, v_{h3,\varepsilon}(\bar{\xi}_i)\} \ . \tag{5.2.30}$$

Step 4 : Obtention of the estimate (5.2.13)

Taking into account that

$$\text{meas}(K) = O(h^2) ,$$

we deduce from finite expansion (5.2.30) that there exists a constant c independent of h such that for any $K \in \mathcal{T}_h$

$$|\tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h) - \gamma^{\alpha}_{\beta}(\vec{v}_h)|_{0,K} \leq c\ h^2 \{ \sum_{j=1}^{3} |v_{hj}|^2_{0,\infty,K} + \sum_{j=1}^{3} |v_{hj}|^2_{1,\infty,K} \}^{1/2} . \quad (5.2.31)$$

The results of interpolation theory in Sobolev spaces show that (see CIARLET [5, theorem 3.1.2])

$$\left. \begin{aligned} |v_{h\alpha}|_{p,\infty,K} &\leq c\ h^{-p}\ |\hat{v}_{h\alpha}|_{p,\infty,\hat{K}} \\ &\leq c\ h^{-p}\ |\hat{v}_{h\alpha}|_{p,\hat{K}} \\ &\leq c\ h^{-1}\ |v_{h\alpha}|_{p,K} \quad , \quad p=0,1 , \end{aligned} \right\} \quad (5.2.32)$$

where the second inequality arises from the result

$$|\hat{v}|_{j,\infty,\hat{K}} \leq c|\hat{v}|_{j,\hat{K}} \quad , \quad j \geq 0 , \quad (5.2.33)$$

available for any polynomial $\hat{v} \in P_k(\hat{K})$, $0 \leq j \leq k$.

Analogously, considering the subtriangles K_i, $K = \bigcup_{i=1}^{3} K_i$, we can prove that

$$|v_{h3}|_{p,\infty,K} \leq c\ h^{-1}\ |v_{h3}|_{p,K} , \quad (5.2.34)$$

so that the relation (5.2.31) gives the estimate (5.2.13). □

Remark 5.2.1 : Since spaces X_{h1} and $\tilde{X}_{h1}$ respectively considered in paragraphs 3.1 and 4.2 are subspaces of corresponding spaces X_{h1} and $\tilde{X}_{h1}$ considered in paragraph 4.4, then the estimate (5.2.13) is still available for the second alternative studied in paragraph 4.4.

□

According to (5.2.10), we have now to estimate $|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K}$.

From the definitions (2.1.12) and (4.2.12), we obtain :

$$\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h) = a^{\alpha\nu}\{v_{h3|\nu\beta} - c_{\nu\beta}v_{h3} + b^{\lambda}_{\beta}|_{\nu}v_{h\lambda} + b^{\lambda}_{\beta}v_{h\lambda|\nu} + b^{\lambda}_{\nu}v_{h\lambda|\beta} \tag{5.2.35}$$

$$\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h) = a^{\alpha\nu}_{h}\tilde{v}_{h3,\nu\beta} \; . \tag{5.2.36}$$

Instead to proceed directly to the study of $|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K}$, we will consider, as an intermediate step, the following theorem which is valid for *general shells without any restriction.* And, after this general result, we will have to introduce some restrictions in order to complete the convergence study

Theorem 5.2.2 : There exists constants $c^{\ell}_{j\alpha\beta}$, $c^{\ell\lambda}_{j\alpha\beta}$, $c^{\varepsilon\eta}_{j\alpha\beta}$, independent of h, such that, for any $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$ and $\vec{v}_h \in \vec{X}_h$ associated through the bijection F_h defined in Theorem 4.2.1, we have :

$$\left.\begin{aligned}
&\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + \\
&\quad + a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi)\sum_{i=1}^{3}A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta} + \\
&\quad (\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}] + O(h)\sum_{k=1}^{3}[c^{\ell}_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k) + \\
&\quad c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\bar{\xi}_k) + c^{\varepsilon\eta}_{j\alpha\beta}v_{h3,\varepsilon\eta}(\bar{\xi}_k)] \; ,
\end{aligned}\right\} \tag{5.2.37}$$

where the expression of $A^{\varepsilon\eta}_{\beta}$ is given by relation (5.2.26).

Proof (in seven steps) :

By definition of space $\tilde{X}_{h2}$, we have on any subtriangle K_j, j=1,2,3, of a given triangle K with vertices $\Sigma_i = (\xi^1_i,\xi^2_i)$, i=1,2,3,

$$\tilde{v}_{h3}(\xi)|_{K_j} = \sum_{i=1}^{3}\tilde{v}_{h3}(\Sigma_i)p^{o}_{j,i}(\lambda) + \sum_{i=1}^{3}D\tilde{v}_{h3}(\Sigma_i)(\Sigma_{i+1}-\Sigma_i)p^{1}_{j,i,i+1}(\lambda)$$
$$+ \sum_{i=1}^{3}D\tilde{v}_{h3}(\Sigma_i)(\Sigma_{i-1}-\Sigma_i)p^{1}_{j,i,i-1}(\lambda) \; ,$$

where $p^o_{j,i}$, $p^1_{j,i,i+1}$, $p^1_{j,i,i-1}$, $1 \le i \le 3$, denote the basis functions for the subtriangle K_j of the reduced finite element of HSIEH-CLOUGH-TOCHER (see BERNADOU-HASSAN [21]) and where $\tilde{v}_{h3|K_j}$ denotes the restriction of function $\tilde{v}_{h3}$ to the subtriangle K_j.

The above expression can be rewritten as

$$\left.\begin{aligned}\tilde{v}_{h3}(\xi)\Big|_{K_j} = \sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i) p^o_{j,i}(\lambda) &+ \sum_{i=1}^{3} (\xi^\nu_{i+1} - \xi^\nu_i) \tilde{v}_{h3,\nu}(\Sigma_i) p^1_{j,i,i+1}(\lambda) \\ &+ \sum_{i=1}^{3} (\xi^\nu_{i-1} - \xi^\nu_i) \tilde{v}_{h3,\nu}(\Sigma_i) p^1_{j,i,i-1}(\lambda)\end{aligned}\right\},$$

so that

$$\left.\begin{aligned}\tilde{v}_{h3,\alpha\beta}(\xi)\Big|_{K_j} &= \sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i)(p^o_{j,i}(\lambda))_{,\alpha\beta} \\ &+ \sum_{i=1}^{3} (\xi^\nu_{i+1} - \xi^\nu_i)\tilde{v}_{h3,\nu}(\Sigma_i)(p^1_{j,i,i+1}(\lambda))_{,\alpha\beta} \\ &+ \sum_{i=1}^{3} (\xi^\nu_{i-1} - \xi^\nu_i)\tilde{v}_{h3,\nu}(\Sigma_i)(p^1_{j,i,i-1}(\lambda))_{,\alpha\beta} .\end{aligned}\right\} \quad (5.2.38)$$

In the following, we give the expressions of $\tilde{v}_{h3}(\Sigma_i)$ and $\tilde{v}_{h3,\nu}(\Sigma_i)$ as functions of degrees of freedom of space $\vec{X}_h$.

<u>Step 1</u> : <u>Expression of</u> $\tilde{v}_{h3}(\Sigma_i)$ <u>as function of degrees of freedom of space</u> $\vec{X}_h$.

By virtue of compatibility relation (4.2.13), we immediatly derive with (5.2.17)

$$\tilde{v}_{h3}(\Sigma_i) = d^j_{h3}(\Sigma_i)\, v_{hj}(\Sigma_i) \; . \quad (5.2.39)$$

Step 2 : Expression of $\tilde{v}_{h3,\nu}(\Sigma_i)$ as functions of degrees of freedom of space $\vec{X}_h$.

From (4.2.26), we have

$$\tilde{v}_{h3,\nu}(\Sigma_i) = e_{\lambda\nu} \sqrt{a_h}\, \tilde{\omega}_h^\lambda(\Sigma_i) \ , \tag{5.2.40}$$

while, from relation (4.2.25) and notation (5.2.17), we obtain $\tilde{\omega}_h^\lambda(\Sigma_i)$ as solutions of the system

$$d_{h\alpha}^\lambda(\Sigma_i)\tilde{\omega}_h^\alpha(\Sigma_i) = B^\lambda(\Sigma_i) - d_{h3}^\lambda(\Sigma_i)\tilde{\omega}_h^3(\Sigma_i) \ . \tag{5.2.41}$$

Let us observe that in the previous equation the expression $\tilde{\omega}_h^3(\Sigma_i)$ is determined as a function of $\vec{v}_h$: indeed, relation (4.2.24) combined with (5.2.18) yields

$$\tilde{\omega}_h^3(\Sigma_i) = \frac{1}{2\sqrt{a_h}}\, e^{\nu\beta} \sum_{k=1}^{3} d_{h\beta}^j(\Sigma_k)\, \frac{\partial\lambda_k}{\partial\xi^\nu}\, v_{hj}(\Sigma_k) \ . \tag{5.2.42}$$

Moreover, from (4.2.22) we have

$$B^\lambda(\Sigma_i) = \frac{1}{\sqrt{a(\Sigma_i)}}\, e^{\lambda\beta} [\, v_{h3,\beta}(\Sigma_i) + b_\beta^\alpha(\Sigma_i)\, v_{h\alpha}(\Sigma_i)] \ . \tag{5.2.43}$$

Then, from equations (5.2.40) (5.2.41), we deduce that $\tilde{v}_{h3,\nu}(\Sigma_i)$ are solutions of the system

$$\left.\begin{aligned} &- \tilde{v}_{h3,1}(\Sigma_i)\, d_{h2}^\lambda(\Sigma_i) + \tilde{v}_{h3,2}(\Sigma_i)\, d_{h1}^\lambda(\Sigma_i) \\ &\qquad = \sqrt{a_h}\, \{B^\lambda(\Sigma_i) - d_{h3}^\lambda(\Sigma_i)\, \tilde{\omega}_h^3(\Sigma_i)\} \ . \end{aligned}\right\}$$

Hence, by denoting

$$d_h(\xi) = d_{h1}^1(\xi)\, d_{h2}^2(\xi) - d_{h1}^2(\xi)\, d_{h2}^1(\xi) \ , \tag{5.2.44}$$

we find

$$\tilde{v}_{h3,\nu}(\Sigma_i) = \frac{e_{\lambda\eta}}{d_h(\Sigma_i)} \; d^{\eta}_{h\nu}(\Sigma_i) \; \sqrt{a_h} \; \{B^{\lambda}(\Sigma_i) - d^{\lambda}_{h3}(\Sigma_i)\tilde{\omega}^3_h(\Sigma_i)\} \; .$$

By combination of this last relation with (5.2.42) (5.2.43), we finally derive :

$$\left.\begin{aligned} \tilde{v}_{h3,\nu}(\Sigma_i) = {} & \frac{\sqrt{a_h}}{\sqrt{a(\Sigma_i)}} \; \frac{d^{\eta}_{h\nu}(\Sigma_i)}{d_h(\Sigma_i)} \; \{b^{\varepsilon}_{\eta}(\Sigma_i) \; v_{h\varepsilon}(\Sigma_i) + v_{h3,\eta}(\Sigma_i)\} \\ & - \frac{1}{2} \, e_{\lambda\eta} \; e^{\mu\varepsilon} \; \frac{d^{\eta}_{h\nu}(\Sigma_i) \; d^{\lambda}_{h3}(\Sigma_i)}{d_h(\Sigma_i)} \; \sum_{k=1}^{3} d^j_{h\varepsilon}(\Sigma_k) \; \frac{\partial\lambda_k}{\partial\xi^{\mu}} \; v_{hj}(\Sigma_k) . \end{aligned}\right\} \quad (5.2.45)$$

Step 3 : Finite expansions of "geometric" coefficients appearing in (5.2.39) and (5.2.45)

We have

$$\left.\begin{aligned} \vec{a}^j(\Sigma_i) = {} & \vec{a}^j(\xi) + (\xi^{\varepsilon}_i - \xi^{\varepsilon})\vec{a}^j_{,\varepsilon}(\xi) + \frac{1}{2}(\xi^{\varepsilon}_i - \xi^{\varepsilon})(\xi^{\eta}_i - \xi^{\eta})\vec{a}^j_{,\varepsilon\eta}(\xi) \\ & + O(h^3)\vec{c}^{\,j}_i \end{aligned}\right\} \quad (5.2.46)$$

where, by denoting

$$\Gamma^3_{\alpha\beta} = b_{\alpha\beta} \quad , \quad \Gamma^{\beta}_{3\alpha} = -b^{\beta}_{\alpha} \quad , \quad \Gamma^3_{3\alpha} = 0 \; , \qquad (5.2.47)$$

and by using (5.2.20) (5.2.21), the derivatives $\vec{a}^j_{,\varepsilon}$ and $\vec{a}^j_{,\varepsilon\eta}$ are given by

$$\left.\begin{aligned} \vec{a}^j_{,\varepsilon} &= - \Gamma^j_{k\varepsilon} \; \vec{a}^k \\ \vec{a}^j_{,\varepsilon\eta} &= \Gamma^j_{i\varepsilon} \; \Gamma^i_{k\eta} \; \vec{a}^k - \Gamma^j_{k\varepsilon,\eta} \; \vec{a}^k \; . \end{aligned}\right\}$$

If we denote

$$S^j_{\varepsilon\eta k}(\xi) = \Gamma^j_{k\varepsilon,\eta}(\xi) - \Gamma^j_{i\varepsilon}(\xi)\Gamma^i_{k\eta}(\xi) \ , \qquad (5.2.48)$$

then, expression (5.2.46) may be rewritten as

$$\left.\begin{aligned}\vec{a}^j(\Sigma_i) = \vec{a}^j(\xi) - (\xi^\varepsilon_i-\xi^\varepsilon)\Gamma^j_{k\varepsilon}(\xi)\vec{a}^k(\xi) - \tfrac{1}{2}(\xi^\varepsilon_i-\xi^\varepsilon)(\xi^\eta_i-\xi^\eta)S^j_{\varepsilon\eta k}(\xi)\vec{a}^k(\xi) \\ + 0(h^3)\vec{c}^j_i \ .\end{aligned}\right\} (5.2.49)$$

Likewise , we have

$$\left.\begin{aligned}\vec{\phi}(\Sigma_i) = \vec{\phi}(\xi) + (\xi^\varepsilon_i-\xi^\varepsilon)\vec{a}_\varepsilon(\xi) + \tfrac{1}{2}(\xi^\varepsilon_i-\xi^\varepsilon)(\xi^\eta_i-\xi^\eta)\vec{a}_{\varepsilon,\eta}(\xi) \\ + \tfrac{1}{6}(\xi^\varepsilon_i-\xi^\varepsilon)(\xi^\eta_i-\xi^\eta)(\xi^\lambda_i-\xi^\lambda)\vec{a}_{\varepsilon,\eta\lambda}(\xi) \\ + 0(h^4)\,\vec{c}_i \ .\end{aligned}\right\}$$

By using (5.2.20) (5.2.21) and (5.2.47), the derivatives $\vec{a}_{j,\varepsilon}$ and $\vec{a}_{j,\varepsilon\eta}$ are given by

$$\vec{a}_{j,\varepsilon}(\xi) = \Gamma^k_{j\varepsilon}(\xi)\,\vec{a}_k(\xi) \ ,$$

$$\vec{a}_{j,\varepsilon\eta}(\xi) = \Gamma^k_{j\varepsilon,\eta}(\xi)\,\vec{a}_k(\xi) + \Gamma^i_{j\varepsilon}(\xi)\Gamma^k_{i\eta}(\xi)\,\vec{a}_k(\xi) \ .$$

Then, since $\vec{a}_{h\beta} = \vec{\phi}_{h,\beta}(\xi) = \sum\limits_{i=1}^{3} \dfrac{\partial\lambda_i}{\partial\xi^\beta}\vec{\phi}(\Sigma_i)$, we get by denoting on the one hand,

$$T^j_{\varepsilon\eta k}(\xi) = \Gamma^j_{k\varepsilon,\eta}(\xi) + \Gamma^i_{k\varepsilon}(\xi)\Gamma^j_{i\eta}(\xi) \ ,$$

and,on the other hand,

$$B^{\varepsilon\eta\lambda}_\beta(\xi) = \frac{1}{6}\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^\beta}(\xi^\varepsilon_i-\xi^\varepsilon)(\xi^\eta_i-\xi^\eta)(\xi^\lambda_i-\xi^\lambda) \ ,$$

and by using (5.2.25) (5.2.26),

$$\left.\begin{aligned} \vec{a}_{h\beta} = \vec{a}_\beta(\xi) &+ A_\beta^{\varepsilon\eta}(\xi)\Gamma_{\varepsilon\eta}^k(\xi)\vec{a}_k(\xi) + B_\beta^{\varepsilon\eta\lambda}(\xi)T_{\eta\lambda\varepsilon}^k(\xi)\vec{a}_k(\xi) \\ &+ 0(h^3)\vec{c}_\beta \ . \end{aligned}\right\} \tag{5.2.50}$$

Let us denote

$$\left.\begin{aligned} D_{\beta 1}^j(\xi,\Sigma_i) &= A_\beta^{\varepsilon\eta}(\xi)\Gamma_{\varepsilon\eta}^j(\xi) - (\xi_i^\varepsilon-\xi^\varepsilon)\Gamma_{\varepsilon\beta}^j(\xi) \\ D_{\beta 2}^j(\xi,\Sigma_i) &= B_\beta^{\varepsilon\eta\lambda}(\xi)T_{\eta\lambda\varepsilon}^j(\xi) - (\xi_i^\varepsilon-\xi^\varepsilon)\Gamma_{\varepsilon k}^j(\xi)\Gamma_{\lambda\eta}^k(\xi)A_\beta^{\lambda\eta}(\xi) \\ &\quad - \frac{1}{2}(\xi_i^\varepsilon-\xi^\varepsilon)(\xi_i^\eta-\xi^\eta)S_{\varepsilon\eta\beta}^j(\xi) \ . \end{aligned}\right\} \tag{5.2.51}$$

Then, from relations (5.2.49) and (5.2.50), we deduce with (5.2.17)

$$d_{h\beta}^j(\Sigma_i) = \delta_\beta^j + D_{\beta 1}^j(\xi,\Sigma_i) + D_{\beta 2}^j(\xi,\Sigma_i) + 0(h^3)c_{\beta i}^j \ . \tag{5.2.52}$$

In order to obtain the finite expansion of $\vec{a}_{h3}$, let us denote

$$\left.\begin{aligned} R_{\beta 1}^k(\xi) &= A_\beta^{\varepsilon\eta}(\xi)\Gamma_{\varepsilon\eta}^k(\xi) \\ R_{\beta 2}^k(\xi) &= B_\beta^{\varepsilon\eta\lambda}(\xi)T_{\eta\lambda\varepsilon}^k(\xi) \end{aligned}\right\} \tag{5.2.53}$$

so that relation (5.2.50) becomes

$$\vec{a}_{h\beta} = (\delta_\beta^k + R_{\beta 1}^k(\xi) + R_{\beta 2}^k(\xi))\vec{a}_k(\xi) + 0(h^3)\vec{c}_\beta \ . \tag{5.2.54}$$

Consequently, since $\vec{a}_{h3} = \dfrac{\vec{a}_{h1} \times \vec{a}_{h2}}{\sqrt{a_h}}$, we obtain

$$\left.\begin{aligned}\vec{a}_{h3} = \sqrt{\frac{a(\xi)}{a_h}}\, \{ & [e_{\alpha\beta}e^{\lambda\mu}R^{\beta}_{\lambda 1}(\xi)R^{3}_{\mu 1}(\xi) - R^{3}_{\alpha 1}(\xi) - R^{3}_{\alpha 2}(\xi)]\vec{a}^{\alpha}(\xi) + \\ & + [1 + R^{\alpha}_{\alpha 1}(\xi) + R^{\alpha}_{\alpha 2}(\xi) + e_{\alpha\beta}R^{\alpha}_{11}(\xi)R^{\beta}_{21}(\xi)]\vec{a}^{3}(\xi) \\ & + 0(h^3)\vec{c}_3\}.\end{aligned}\right\} \quad (5.2.55)$$

This finite expansion combined with (5.2.49) gives

$$d^{\nu}_{h3}(\Sigma_i) = \sqrt{\frac{a(\xi)}{a_h}}\,\{D^{\nu}_{31}(\xi,\Sigma_i) + D^{\nu}_{32}(\xi,\Sigma_i) + 0(h^3)c^{\nu}_{3i}\} \quad (5.2.56)$$

where we have set

$$\left.\begin{aligned} D^{\nu}_{31}(\xi,\Sigma_i) &= -R^{3}_{\alpha 1}(\xi)a^{\nu\alpha}(\xi) - (\xi^{\varepsilon}_i-\xi^{\varepsilon})\Gamma^{\nu}_{\varepsilon 3}(\xi) \ , \\ D^{\nu}_{32}(\xi,\Sigma_i) &= [e_{\alpha\beta}e^{\lambda\mu}R^{\beta}_{\lambda 1}(\xi)R^{3}_{\mu 1}(\xi) - R^{3}_{\alpha 2}(\xi)]a^{\alpha\nu}(\xi) \\ &\quad + (\xi^{\varepsilon}_i-\xi^{\varepsilon})R^{3}_{\alpha 1}(\xi)\Gamma^{\nu}_{\varepsilon\beta}(\xi)a^{\alpha\beta}(\xi) \\ &\quad - R^{\alpha}_{\alpha 1}(\xi)(\xi^{\varepsilon}_i-\xi^{\varepsilon})\Gamma^{\nu}_{\varepsilon 3}(\xi) - \tfrac{1}{2}(\xi^{\varepsilon}_i-\xi^{\varepsilon})(\xi^{\eta}_i-\xi^{\eta})s^{\nu}_{\varepsilon\eta 3}(\xi) \ . \end{aligned}\right\} \quad (5.2.57)$$

Likewise, upon combining (5.2.49) and (5.2.55), we deduce

$$d^{3}_{h3}(\Sigma_i) = \sqrt{\frac{a(\xi)}{a_h}}\,\{1 + D^{3}_{31}(\xi,\Sigma_i) + D^{3}_{32}(\xi,\Sigma_i) + 0(h^3)c^{3}_{3i}\} \quad (5.2.58)$$

where we have denoted

$$\left.\begin{aligned} D^{3}_{31}(\xi,\Sigma_i) &= R^{\alpha}_{\alpha 1}(\xi) \\ D^{3}_{32}(\xi,\Sigma_i) &= R^{\alpha}_{\alpha 2}(\xi) + e_{\alpha\beta}R^{\alpha}_{11}(\xi)R^{\beta}_{21}(\xi) + (\xi^{\varepsilon}_i-\xi^{\varepsilon})R^{3}_{\alpha 1}(\xi)b^{\alpha}_{\varepsilon}(\xi) \\ &\quad - \frac{1}{2}(\xi^{\varepsilon}_i-\xi^{\varepsilon})(\xi^{\eta}_i-\xi^{\eta})s^{3}_{\varepsilon\eta 3}(\xi) \ . \end{aligned}\right\} \quad (5.2.59)$$

From (5.2.54), we get

$$\left.\begin{aligned} a_{h\alpha\beta} = a_{\alpha\beta}(\xi) &+ \{R^{\lambda}_{\alpha 1}(\xi) + R^{\lambda}_{\alpha 2}(\xi)\}a_{\beta\lambda}(\xi) + \{R^{\lambda}_{\beta 1}(\xi) + R^{\lambda}_{\beta 2}(\xi)\}a_{\alpha\lambda}(\xi) \\ &+ R^{i}_{\alpha 1}(\xi)R^{j}_{\beta 1}(\xi)a_{ij}(\xi) + 0(h^3)c_{\alpha\beta} \ , \end{aligned}\right\}$$

and thus

$$a_h = \{1 + 2R^{\lambda}_{\lambda 1}(\xi) + R(\xi)\}a(\xi) + 0(h^3)c , \tag{5.2.60}$$

where we denote

$$\left.\begin{aligned} R(\xi) &= 2R^{\lambda}_{\lambda 2}(\xi) + R^1_{11}(\xi)R^1_{11}(\xi) + R^2_{21}(\xi)R^2_{21}(\xi) \\ &\quad + 4R^1_{11}(\xi)R^2_{21}(\xi) - 2R^2_{11}(\xi)R^1_{21}(\xi) \\ &\quad + \frac{1}{a(\xi)} e^{\alpha\beta}e^{\lambda\mu}R^3_{\alpha 1}(\xi)R^3_{\lambda 1}(\xi)a_{\beta\mu}(\xi) . \end{aligned}\right\}$$

From this last finite expansion, we find

$$\sqrt{\frac{a(\xi)}{a_h}} = 1 - R^{\lambda}_{\lambda 1}(\xi) + \frac{3}{2}[R^{\lambda}_{\lambda 1}(\xi)]^2 - \frac{1}{2}R(\xi) + 0(h^3)c , \tag{5.2.61}$$

and thus, we obtain with (5.2.56) and (5.2.58) :

$$\left.\begin{aligned} d^{\nu}_{h3}(\Sigma_i) &= D^{\nu}_{31}(\xi,\Sigma_i) - R^{\lambda}_{\lambda 1}(\xi)D^{\nu}_{31}(\xi,\Sigma_i) + D^{\nu}_{32}(\xi,\Sigma_i) + 0(h^3)c^{\nu}_{3i} , \\ d^3_{h3}(\Sigma_i) &= 1 + D^3_{31}(\xi,\Sigma_i) - R^{\lambda}_{\lambda 1}(\xi) + D^3_{32}(\xi,\Sigma_i) - R^{\lambda}_{\lambda 1}(\xi)D^3_{31}(\xi,\Sigma_i) \\ &\quad + \frac{3}{2}[R^{\lambda}_{\lambda 1}(\xi)]^2 - \frac{1}{2}R(\xi) + 0(h^3)c^3_{3i} \\ &= 1 + D^3_{32}(\xi,\Sigma_i) + \frac{1}{2}(R^{\lambda}_{\lambda 1}(\xi))^2 - \frac{1}{2}R(\xi) + 0(h^3)c^3_{3i} . \end{aligned}\right\} \tag{5.2.62}$$

Finally, the finite expansion (5.2.52) gives with (5.2.44)

$$\left.\begin{aligned} d_h(\Sigma_i) &= 1 + D^1_{11}(\xi,\Sigma_i) + D^2_{21}(\xi,\Sigma_i) + D^2_{22}(\xi,\Sigma_i) + D^1_{12}(\xi,\Sigma_i) \\ &\quad + D^1_{11}(\xi,\Sigma_i)D^2_{21}(\xi,\Sigma_i) - D^1_{21}(\xi,\Sigma_i)D^2_{11}(\xi,\Sigma_i) + 0(h^3)c . \end{aligned}\right\} \tag{5.2.63}$$

<u>Step 4</u> : <u>Finite expansion of</u> $\tilde{v}_{h3}(\Sigma_i)$

Upon combining the finite expansion (5.2.62) with (5.2.39), we obtain

$$\left.\begin{aligned}\tilde{v}_{h3}(\Sigma_i) &= v_{h3}(\Sigma_i) + \{[1 - R^{\lambda}_{\lambda 1}(\xi)]D^{\nu}_{31}(\xi,\Sigma_i) + D^{\nu}_{32}(\xi,\Sigma_i)\}v_{h\nu}(\Sigma_i) \\ &+ \frac{1}{2}\{2D^{3}_{32}(\xi,\Sigma_i) + (R^{\lambda}_{\lambda 1}(\xi))^2 - R(\xi)\}v_{h3}(\Sigma_i) + \\ &+ 0(h^3)c^{j}_{3i}v_{hj}(\Sigma_i)\ .\end{aligned}\right\} \quad (5.2.64)$$

Step 5 : Finite expansion of $\tilde{v}_{h3,\nu}(\Sigma_i)$

Combining finite expansions (5.2.52), (5.2.61), (5.2.62), (5.2.63) with relation (5.2.45), we get the existence of constants $c^{\varepsilon}_{3\nu}$, c^{λ}_{ν} , c^{j}_{ν} , independent of h, such that

$$\left.\begin{aligned}&\tilde{v}_{h3,\nu}(\Sigma_i) = v_{h3,\nu}(\Sigma_i) + b^{\lambda}_{\nu}(\Sigma_i)v_{h\lambda}(\Sigma_i) \\ &+ D^{\lambda}_{\nu 1}(\xi,\Sigma_i)(v_{h3,\lambda}(\Sigma_i) + b^{\varepsilon}_{\lambda}(\Sigma_i)v_{h\varepsilon}(\Sigma_i)) \\ &+ \frac{1}{2}e_{\gamma\nu}\{(\xi^{\varepsilon}_i-\xi^{\varepsilon})b^{\gamma}_{\varepsilon}(\xi) - b_{\varepsilon\eta}(\xi)a^{\gamma\lambda}(\xi)A^{\varepsilon\eta}_{\lambda}(\xi)\}(v_{h1,2}(\xi)-v_{h2,1}(\xi)) \\ &+ 0(h^2)\{c^{\varepsilon}_{3\nu}v_{h3,\varepsilon}(\Sigma_i) + c^{\omega}v_{h\omega,\nu} + \sum_{k=1}^{3}[c^{j}_{\nu}v_{hj}(\Sigma_k) + c^{\lambda}_{\nu}v_{h3,\lambda}(\bar{\xi}_k)]\}\ ,\end{aligned}\right\}(5.2.65)$$

where $\bar{\xi}_k \in [\xi,\Sigma_k]$.

Step 6 : Finite expansion of $\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)$

This finite expansion is obtained by substituting (5.2.64) and (5.2.65) into the expression (5.2.38). Then, we get the existence of constants $c^{j}_{\alpha\beta}$, $c^{\lambda}_{\alpha\beta}$, independent of h, such that at any point ξ of any subtriangle K_j, j = 1,2,3, of the triangle K with vertices Σ_i , i=1,2,3, we have

$$\left.\begin{aligned}
\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\Big|_{K_j} &= \sum_{i=1}^{3} \{[[1-R^{\lambda}_{\lambda 1}(\xi)]D^{\nu}_{31}(\xi,\Sigma_i) + D^{\nu}_{32}(\xi,\Sigma_i)]v_{h\nu}(\Sigma_i) \\
&+ [1 + D^{3}_{32}(\xi,\Sigma_i) + \frac{1}{2}(R^{\lambda}_{\lambda 1}(\xi))^2 - \frac{1}{2}R(\xi)]v_{h3}(\Sigma_i)\}(p^{o}_{j,i})_{,\alpha\beta} \\
&+ \sum_{i=1}^{3} \{v_{h3,\nu}(\Sigma_i) + b^{\lambda}_{\nu}(\Sigma_i)v_{h\lambda}(\Sigma_i) \\
&+ \frac{1}{2} e_{\gamma\nu}[(\xi^{\varepsilon}_i-\xi^{\varepsilon})b^{\gamma}_{\varepsilon}(\xi) - b_{\varepsilon\eta}(\xi)a^{\gamma\lambda}(\xi)A^{\varepsilon\eta}_{\lambda}(\xi)](v_{h1,2}(\xi) - v_{h2,1}(\xi)) \\
&+ D^{\lambda}_{\nu 1}(\xi,\Sigma_i)[v_{h3,\lambda}(\Sigma_i) + b^{\varepsilon}_{\lambda}(\Sigma_i)v_{h\varepsilon}(\Sigma_i)]\} \\
&\quad \times \{(\xi^{\nu}_{i+1}-\xi^{\nu}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta}\} \\
&+ O(h)\sum_{k=1}^{3}[c^{\ell}_{j\alpha\beta}v_{h\ell}(\Sigma_k) + c^{\lambda}_{j\alpha\beta}v_{h3,\lambda}(\Sigma_k) + c^{\lambda}_{j\alpha\beta}v_{h3,\lambda}(\bar{\xi}_k)] ,
\end{aligned}\right\} \quad (5.2.66)$$

where $\bar{\xi}_k \in [\xi,\Sigma_k]$.

Now, let us study this finite expansion. Let us begin by proving some characteristic relations.

First, if we consider the function $F(\xi) = 1$ and if we write this function on the basis $\{p^{o}_{j,i} , p^{1}_{j,i,i+1} , p^{1}_{j,i,i-1}\}$, we obtain

$$1 = \sum_{i=1}^{3} p^{o}_{j,i}(\lambda) ,$$

so that we deduce the relation

$$\sum_{i=1}^{3} (p^{o}_{j,i})_{,\alpha\beta} = 0 . \qquad (5.2.67)$$

In the same way, let us consider the function $G^{\varepsilon}(\xi) = \xi^{\varepsilon}$ and let us write this function on the basis $\{p^{o}_{j,i} , p^{1}_{j,i,i+1} , p^{1}_{j,i,i-1}\}$. Since $G^{\varepsilon}_{,\alpha\beta}(\xi) = 0$, we obtain

$$\left.\begin{aligned}
&\sum_{i=1}^{3} \xi^{\varepsilon}_i(p^{o}_{j,i})_{,\alpha\beta} + \sum_{i=1}^{3} (\xi^{\varepsilon}_{i+1}-\xi^{\varepsilon}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} \\
&\qquad + \sum_{i=1}^{3} (\xi^{\varepsilon}_{i-1}-\xi^{\varepsilon}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta} = 0 .
\end{aligned}\right\} \quad (5.2.68)$$

Next, let us consider the function $F^{\varepsilon\eta}(\xi) = \frac{1}{2}\,\xi^{\varepsilon}\xi^{\eta}$ and let us write this function on the basis $\{\overset{o}{p}_{j,i}\ ,\ \overset{1}{p}_{j,i,i+1}\ ,\ \overset{1}{p}_{j,i,i-1}\}$.

Since $F^{\varepsilon\eta}{}_{,\alpha\beta}(\xi) = \frac{1}{2}\,(\delta^{\varepsilon}_{\alpha}\delta^{\eta}_{\beta} + \delta^{\eta}_{\alpha}\delta^{\varepsilon}_{\beta})$, then, by using (5.2.67) and (5.2.68), we easily deduce

$$\left.\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{3}(\xi^{\varepsilon}_{i}-\xi^{\varepsilon})(\xi^{\eta}_{i}-\xi^{\eta})(\overset{o}{p}_{j,i})_{,\alpha\beta}\\
&+\frac{1}{2}\sum_{i=1}^{3}(\xi^{\varepsilon}_{i}-\xi^{\varepsilon})\{(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(\overset{1}{p}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(\overset{1}{p}_{j,i,i-1})_{,\alpha\beta}\}\\
&+\frac{1}{2}\sum_{i=1}^{3}(\xi^{\eta}_{i}-\xi^{\eta})\{(\xi^{\varepsilon}_{i+1}-\xi^{\varepsilon}_{i})(\overset{1}{p}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\varepsilon}_{i-1}-\xi^{\varepsilon}_{i})(\overset{1}{p}_{j,i,i-1})_{,\alpha\beta}\}\\
&=\frac{1}{2}\,(\delta^{\varepsilon}_{\alpha}\delta^{\eta}_{\beta} + \delta^{\eta}_{\alpha}\delta^{\varepsilon}_{\beta})\ .
\end{aligned}\right\}\quad(5.2.69)$$

Finally, if we set

$$G^{\varepsilon\eta}(\xi) = -\frac{1}{2}\,\{\xi^{\varepsilon}\xi^{\eta} - \sum_{k=1}^{3}\lambda_k(\xi)\xi^{\varepsilon}_{k}\xi^{\eta}_{k}\}\quad ,$$

we observe that

$$\left.\begin{aligned}
&G^{\varepsilon\eta}(\Sigma_i) = 0\\
&G^{\varepsilon\eta}{}_{,\beta}(\xi) = A^{\varepsilon\eta}_{\beta}(\xi)\qquad (\text{see } (5.2.26))\\
&G^{\varepsilon\eta}{}_{,\alpha\beta}(\xi) = -\frac{1}{2}\,(\delta^{\varepsilon}_{\alpha}\delta^{\eta}_{\beta} + \delta^{\varepsilon}_{\beta}\delta^{\eta}_{\alpha})\ .
\end{aligned}\right\}\qquad(5.2.70)$$

Since $G^{\varepsilon\eta}$ is polynomial of degree 2, this function is invariant by HCT-R interpolation. Thus, we find with (5.2.70)

$$\left.\begin{aligned}
&\sum_{i=1}^{3} A^{\varepsilon\eta}_{\lambda}(\Sigma_i)\{(\xi^{\lambda}_{i+1}-\xi^{\lambda}_{i})(\overset{1}{p}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\lambda}_{i-1}-\xi^{\lambda}_{i})(\overset{1}{p}_{j,i,i-1})_{,\alpha\beta}\}\\
&\qquad = -\frac{1}{2}\,(\delta^{\varepsilon}_{\alpha}\delta^{\eta}_{\beta} + \delta^{\varepsilon}_{\beta}\delta^{\eta}_{\alpha})\ .
\end{aligned}\right\}\quad(5.2.71)$$

Now, we are able to analyze the finite expansion (5.2.66).

Firs, let us observe that, by using (5.2.57) (5.2.59) and (5.2.67), the coefficient of $(\overset{o}{p}_{j,i})_{,\alpha\beta}$ in the relation (5.2.66) can be rewritten as :

$$\left.\begin{aligned}
&\sum_{i=1}^{3} \{[[1-R^{\lambda}_{\lambda 1}(\xi)]D^{\nu}_{31}(\xi,\Sigma_i) + D^{\nu}_{32}(\xi,\Sigma_i)]v_{h\nu}(\Sigma_i) \\
&+ [1 + D^{3}_{32}(\xi,\Sigma_i) + \tfrac{1}{2}(R^{\lambda}_{\lambda 1}(\xi))^2 - \tfrac{1}{2}R(\xi)]v_{h3}(\Sigma_i)\}(p^{o}_{j,i})_{,\alpha\beta} \\
&= \sum_{i=1}^{3} (\xi^{\varepsilon}_i-\xi^{\varepsilon})\{b^{\nu}_{\varepsilon}(\xi) + a^{\lambda\mu}(\xi)R^{3}_{\lambda 1}(\xi)\Gamma^{\nu}_{\varepsilon\mu}(\xi) \\
&+ \tfrac{1}{2}(\xi^{\eta}_i-\xi^{\eta})[b^{\nu}_{\varepsilon,\eta}(\xi)-b^{\lambda}_{\eta}(\xi)\Gamma^{\nu}_{\lambda\varepsilon}(\xi)]\}(p^{o}_{j,i})_{,\alpha\beta}v_{h\nu}(\xi) \\
&+ \sum_{i=1}^{3} (\xi^{\varepsilon}_i-\xi^{\varepsilon})\{b^{\nu}_{\eta}(\xi)(\xi^{\eta}_i-\xi^{\eta}) - A^{\lambda\eta}_{\mu}(\xi)b_{\lambda\eta}(\xi)a^{\nu\mu}(\xi)\}(p^{o}_{j,i})_{,\alpha\beta}v_{h\nu,\varepsilon} \\
&+ \sum_{i=1}^{3} (p^{o}_{j,i})_{,\alpha\beta}v_{h3}(\Sigma_i) \\
&+ \sum_{i=1}^{3} (\xi^{\varepsilon}_i-\xi^{\varepsilon})\{b^{\lambda}_{\varepsilon}(\xi)R^{3}_{\lambda 1}(\xi) - \tfrac{1}{2}(\xi^{\eta}_i-\xi^{\eta})b^{\lambda}_{\varepsilon}(\xi)b_{\lambda\eta}(\xi)\}(p^{o}_{j,i})_{,\alpha\beta}v_{h3}(\xi) \\
&+ O(h)\sum_{i=1}^{3} c^{k\varepsilon}_{j\alpha\beta}v_{hk,\varepsilon}(\bar{\xi}_i) \\
&\text{where } \bar{\xi}_i \in [\xi,\Sigma_i]\ .
\end{aligned}\right\} \quad (5.2.72)$$

Next, we have

$$\left.\begin{aligned}
&\sum_{i=1}^{3} b^{\lambda}_{\nu}(\Sigma_i)v_{h\lambda}(\Sigma_i)\{(\xi^{\nu}_{i+1}-\xi^{\nu}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta}\} \\
&= \sum_{i=1}^{3} \{(\xi^{\nu}_{i+1}-\xi^{\nu}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta}\}b^{\lambda}_{\nu}(\xi)v_{h\lambda}(\xi) \\
&+ \sum_{i=1}^{3} (\xi^{\varepsilon}_i-\xi^{\varepsilon})\{(\xi^{\nu}_{i+1}-\xi^{\nu}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta}\} \\
&\qquad \times \{b^{\lambda}_{\nu}(\xi)v_{h\lambda,\varepsilon}(\xi) + b^{\lambda}_{\nu,\varepsilon}(\xi)v_{h\lambda}(\xi)\} \\
&+ O(h)\sum_{k=1}^{3} [c^{\lambda}_{j\alpha\beta}v_{h\lambda}(\bar{\xi}_k) + c^{\lambda\varepsilon}_{j\alpha\beta}v_{h\lambda,\varepsilon}(\hat{\xi}_k)]\ .
\end{aligned}\right\} \quad (5.2.73)$$

But, by definition of space X_{h2}, we have (refer to Step 1 and (5.2.38)) :

$$\left.\begin{aligned}
&v_{h3,\alpha\beta}(\xi)\Big|_{K_j} = \sum_{i=1}^{3} (p^{o}_{j,i})_{,\alpha\beta}v_{h3}(\Sigma_i) \\
&\quad + \sum_{i=1}^{3} \{(\xi^{\nu}_{i+1}-\xi^{\nu}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta}\}v_{h3,\nu}(\Sigma_i)
\end{aligned}\right\} \quad (5.2.74)$$

Next, on the one hand, we obtain by using (5.2.67) (5.2.68)

$$\left.\begin{aligned}\{\sum_{i=1}^{3} (\xi_i^{\varepsilon}-\xi^{\varepsilon})(p_{j,i}^{o})_{,\alpha\beta} + \sum_{i=1}^{3} \{(\xi_{i+1}^{\varepsilon}-\xi_i^{\varepsilon})(p_{j,i,i+1}^{1})_{,\alpha\beta}\\ + (\xi_{i-1}^{\varepsilon}-\xi^{\varepsilon})(p_{j,i,i-1}^{1})_{,\alpha\beta}\}\}b_{\varepsilon}^{\nu}(\xi)v_{h\nu}(\xi) = 0\end{aligned}\right\} \quad (5.2.75)$$

and, on the other hand, by using (5.2.69), we deduce

$$\left.\begin{aligned}&\{\sum_{i=1}^{3} (\xi_i^{\varepsilon}-\xi^{\varepsilon})(\xi_i^{\eta}-\xi^{\eta})(p_{j,i}^{o})_{,\alpha\beta}\\ &+ \sum_{i=1}^{3} (\xi_i^{\eta}-\xi^{\eta})[(\xi_{i+1}^{\varepsilon}-\xi_i^{\varepsilon})(p_{j,i,i+1}^{1})_{,\alpha\beta} + (\xi_{i-1}^{\varepsilon}-\xi_i^{\varepsilon})(p_{j,i,i-1}^{1})_{,\alpha\beta}]\\ &+ \sum_{i=1}^{3} (\xi_i^{\varepsilon}-\xi^{\varepsilon})[(\xi_{i+1}^{\eta}-\xi_i^{\eta})(p_{j,i,i+1}^{1})_{,\alpha\beta} + (\xi_{i-1}^{\eta}-\xi_i^{\eta})(p_{j,i,i-1}^{1})_{,\alpha\beta}]\}\\ &\qquad \times [b_{\eta}^{\lambda}(\xi)v_{h\lambda,\varepsilon}(\xi) + \frac{1}{2} b_{\eta,\varepsilon}^{\lambda}(\xi)v_{h\lambda}(\xi)]\\ &= b_{\beta}^{\lambda}(\xi)v_{h\lambda,\alpha}(\xi) + \frac{1}{2} b_{\beta,\alpha}^{\lambda}(\xi)v_{h\lambda}(\xi)\\ &+ b_{\alpha}^{\lambda}(\xi)v_{h\lambda,\beta}(\xi) + \frac{1}{2} b_{\alpha,\beta}^{\lambda}(\xi)v_{h\lambda}(\xi) .\end{aligned}\right\} (5.2.76)$$

Also, note that we can write

$$A_{\nu}^{\varepsilon\eta}(\xi) = A_{\nu}^{\varepsilon\eta}(\Sigma_i) + \frac{1}{2} (\delta_{\nu}^{\varepsilon}\delta_{\lambda}^{\eta} + \delta_{\lambda}^{\varepsilon}\delta_{\nu}^{\eta})(\xi_i^{\lambda}-\xi^{\lambda}) \quad (5.2.77)$$

so that

$$D_{\nu 1}^{\lambda}(\xi,\Sigma_i) = A_{\nu}^{\varepsilon\eta}(\Sigma_i)\Gamma_{\varepsilon\eta}^{\lambda}(\xi)$$

Finally, from the above relation, (5.2.51) and (5.2.71), we obtain

$$\left.\begin{aligned}&\sum_{i=1}^{3} D_{\nu 1}^{\lambda}(\xi,\Sigma_i)[v_{h3,\lambda}(\Sigma_i) + b_{\lambda}^{\varepsilon}(\Sigma_i)v_{h\varepsilon}(\Sigma_i)]\{(\xi_{i+1}^{\nu}-\xi_i^{\nu})(p_{j,i,i+1}^{1})_{,\alpha\beta}\\ &\qquad + (\xi_{i-1}^{\nu}-\xi_i^{\nu})(p_{j,i,i-1}^{1})_{,\alpha\beta}\\ &= - \Gamma_{\alpha\beta}^{\lambda}(\xi)[v_{h3,\lambda}(\xi) + b_{\lambda}^{\nu}(\xi)v_{h\nu}(\xi)]\\ &+ O(h) \sum_{k=1}^{3} [c_{j\alpha\beta}^{\varepsilon\eta}v_{h3,\varepsilon\eta}(\bar{\xi}_k) + c_{j\alpha\beta}^{\nu\varepsilon}v_{h\nu,\varepsilon}(\bar{\xi}_k) + c_{j\alpha\beta}^{\varepsilon}v_{h\varepsilon}(\bar{\xi}_k)]\end{aligned}\right\} (5.2.78)$$

Then, combining relations (5.2.72) to (5.2.76) and (5.2.78) with finite expansion (5.2.66), we get the existence of constants $c^{\ell}_{j\alpha\beta}$, $c^{\ell\lambda}_{j\alpha\beta}$, $c^{\varepsilon\eta}_{j\alpha\beta}$ independent of h such that

$$\left.\begin{aligned}
&\tilde{\rho}_{h\alpha\beta}(\tilde{\vec{v}}_h)\big|_{K_j} = v_{h3,\alpha\beta}(\xi) + \frac{1}{2}\,[b^{\lambda}_{\alpha,\beta}(\xi)+b^{\lambda}_{\alpha,\beta}(\xi)]v_{h\lambda}(\xi)\\
&+ b^{\lambda}_{\alpha}(\xi)v_{h\lambda,\beta}(\xi) + b^{\lambda}_{\beta}(\xi)v_{h\lambda,\alpha}(\xi) - \Gamma^{\lambda}_{\alpha\beta}(\xi)[v_{h3,\lambda}(\xi)+b^{\nu}_{\lambda}(\xi)v_{h\nu}(\xi)]\\
&+ \sum_{i=1}^{3}\{(\xi^{\varepsilon}_i-\xi^{\varepsilon})[R^3_{\lambda 1}(\xi)\Gamma^{\nu}_{\varepsilon\mu}(\xi)a^{\lambda\mu}(\xi) - \frac{1}{2}\,(\xi^{\eta}_i-\xi^{\eta})b^{\lambda}_{\eta}(\xi)\Gamma^{\nu}_{\lambda\varepsilon}(\xi)]\}(p^{o}_{j,i})_{,\alpha\beta}v_{h\nu}(\xi)\\
&+ \sum_{i=1}^{3}\{(\xi^{\varepsilon}_i-\xi^{\varepsilon})[R^3_{\lambda 1}(\xi)b^{\lambda}_{\varepsilon}(\xi) - \frac{1}{2}\,(\xi^{\eta}_i-\xi^{\eta})b^{\lambda}_{\varepsilon}(\xi)b_{\lambda\eta}(\xi)]\}(p^{o}_{j,i})_{,\alpha\beta}v_{h3}(\xi)\\
&- \sum_{i=1}^{3}(\xi^{\varepsilon}_i-\xi^{\varepsilon})A^{\lambda\eta}_{\mu}(\xi)b_{\lambda\eta}(\xi)a^{\nu\mu}(\xi)(p^{o}_{j,i})_{,\alpha\beta}v_{h\nu,\varepsilon}(\xi)\\
&+ \sum_{i=1}^{3}\{(\xi^{\nu}_{i+1}-\xi^{\nu}_i)(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_i)(p^{1}_{j,i,i-1})_{,\alpha\beta}\}\times\\
&\qquad\times\{\frac{1}{2}\,e_{\gamma\nu}[(\xi^{\varepsilon}_i-\xi^{\varepsilon})b^{\gamma}_{\varepsilon}(\xi)-b_{\varepsilon\eta}(\xi)a^{\gamma\lambda}(\xi)A^{\varepsilon\eta}_{\lambda}(\xi)](v_{h1,2}(\xi)-v_{h2,1}(\xi))\\
&\qquad+ (\xi^{\varepsilon}_i-\xi^{\varepsilon})\,[\frac{1}{2}\,(b^{\lambda}_{\nu,\varepsilon}(\xi)-b^{\lambda}_{\varepsilon,\nu}(\xi))v_{h\lambda}(\xi) - b^{\lambda}_{\varepsilon}(\xi)v_{h\lambda,\nu}(\xi)]\}\\
&+ O(h)\sum_{i=1}^{3}[c^{\ell}_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k) + c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\bar{\xi}_k) + c^{\varepsilon\eta}_{j\alpha\beta}v_{h3,\varepsilon\eta}(\bar{\xi}_k)]
\end{aligned}\right\}\quad(5.2.79)$$

where the points $\bar{\xi}_k \in [\xi,\Sigma_k]$; they can change from an expression to the next, as well as the values of constants $c^{\ell}_{j\alpha\beta}$, ...

<u>Step 7</u> : <u>Analysis of finite expansion</u> (5.2.79)

By considering the different terms which appear at the second member of equation (5.2.79), we get successively :

$$v_{h3,\alpha\beta}(\xi) - \Gamma^{\lambda}_{\alpha\beta}(\xi)v_{h3,\lambda}(\xi) = v_{h3|\alpha\beta}(\xi) \qquad (5.2.80)$$

$$\left.\begin{aligned}
&\frac{1}{2}\,[b^{\lambda}_{\alpha,\beta}(\xi)+b^{\lambda}_{\beta,\alpha}(\xi)]v_{h\lambda}(\xi) + b^{\lambda}_{\alpha}(\xi)v_{h\lambda,\beta}(\xi) \\
&\quad + b^{\lambda}_{\beta}(\xi)v_{h\lambda,\alpha}(\xi) - \Gamma^{\lambda}_{\alpha\beta}(\xi)b^{\nu}_{\lambda}(\xi)v_{h\nu}(\xi) \\
&\quad = b^{\lambda}_{\alpha}\big|_{\beta}(\xi)v_{h\lambda}(\xi) + b^{\lambda}_{\alpha}(\xi)v_{h\lambda|\beta}(\xi) + b^{\lambda}_{\beta}(\xi)v_{h\lambda|\alpha}(\xi) \\
&\quad + \frac{1}{2}\,(\Gamma^{\lambda}_{\nu\beta}(\xi)b^{\nu}_{\alpha}(\xi)+\Gamma^{\lambda}_{\nu\alpha}(\xi)b^{\nu}_{\beta}(\xi))v_{h\lambda}(\xi) \;.
\end{aligned}\right\} \qquad (5.2.81)$$

By using relation (5.2.69) we find

$$\left.\begin{aligned}
&-\frac{1}{2}\,b^{\lambda}_{\varepsilon}(\xi)b_{\lambda\eta}(\xi)v_{h3}(\xi)\sum_{i=1}^{3}(\xi^{\varepsilon}_{i}-\xi^{\varepsilon})(\xi^{\eta}_{i}-\xi^{\eta})(p^{o}_{j,i})_{,\alpha\beta} \\
&\quad = -\,b^{\lambda}_{\alpha}(\xi)b_{\lambda\beta}(\xi)v_{h3}(\xi) \\
&\quad + b^{\lambda}_{\varepsilon}(\xi)b_{\lambda\eta}(\xi)v_{h3}(\xi)\sum_{i=1}^{3}(\xi^{\varepsilon}_{i}-\xi^{\varepsilon})[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta} + \\
&\quad + (\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}]
\end{aligned}\right\} \qquad (5.2.82)$$

Since $b^{\lambda}_{\eta}\big|_{\varepsilon}(\xi) = b^{\lambda}_{\varepsilon}\big|_{\eta}(\xi)$, we obtain

$$b^{\lambda}_{\eta,\varepsilon}(\xi) - b^{\lambda}_{\varepsilon,\eta}(\xi) = \Gamma^{\lambda}_{\eta\mu}(\xi)b^{\mu}_{\varepsilon}(\xi) - \Gamma^{\lambda}_{\varepsilon\mu}(\xi)b^{\mu}_{\eta}(\xi)$$

so that, with relation (5.2.69), we have

$$
\left.\begin{aligned}
&-\frac{1}{2}\, b^{\lambda}_{\eta}(\xi)\Gamma^{\nu}_{\lambda\varepsilon}(\xi)v_{h\nu}(\xi) \sum_{i=1}^{3} (\xi^{\varepsilon}_{i}-\xi^{\varepsilon})(\xi^{\eta}_{i}-\xi^{\eta})(p^{o}_{j,i})_{,\alpha\beta} \\
&+\frac{1}{2}\,(b^{\lambda}_{\eta,\varepsilon}(\xi)-b^{\lambda}_{\varepsilon,\eta}(\xi))v_{h\lambda}(\xi)\ \times \\
&\qquad \times \sum_{i=1}^{3} (\xi^{\varepsilon}_{i}-\xi^{\varepsilon})[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}] \\
&= -\frac{1}{2}\,\{b^{\lambda}_{\beta}(\xi)\Gamma^{\nu}_{\lambda\alpha}(\xi) + b^{\lambda}_{\alpha}(\xi)\Gamma^{\nu}_{\lambda\beta}(\xi)\}v_{h\nu}(\xi) \\
&+\frac{1}{2}\,(b^{\nu}_{\eta}(\xi)\Gamma^{\lambda}_{\nu\varepsilon}(\xi)+b^{\nu}_{\varepsilon}(\xi)\Gamma^{\lambda}_{\nu\eta}(\xi))v_{h\lambda}(\xi)\ \times \\
&\qquad \times \sum_{i=1}^{3} (\xi^{\varepsilon}_{i}-\xi^{\varepsilon})[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}] \\
&+\frac{1}{2}\,(b^{\nu}_{\varepsilon}(\xi)\Gamma^{\lambda}_{\nu\eta}(\xi)-b^{\nu}_{\eta}(\xi)\Gamma^{\lambda}_{\varepsilon\nu}(\xi))v_{h\lambda}(\xi)\ \times \\
&\qquad \times \sum_{i=1}^{3} (\xi^{\varepsilon}_{i}-\xi^{\varepsilon})[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}] \\
&= -\frac{1}{2}\,\{b^{\lambda}_{\beta}(\xi)\Gamma^{\nu}_{\lambda\alpha}(\xi) + b^{\lambda}_{\alpha}(\xi)\Gamma^{\nu}_{\lambda\beta}(\xi)\}v_{h\nu}(\xi) \\
&+ b^{\nu}_{\varepsilon}(\xi)\Gamma^{\lambda}_{\nu\eta}(\xi)v_{h\lambda}(\xi)\ \times \\
&\qquad \times \sum_{i=1}^{3} (\xi^{\varepsilon}_{i}-\xi^{\varepsilon})[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}]
\end{aligned}\right\} \quad (5.2.83)
$$

Since $R^{3}_{\mu 1}(\xi) = A^{\lambda\eta}_{\mu}(\xi)b_{\lambda\eta}(\xi)$ and

$$\bar{\rho}_{\alpha\beta}(\vec{v}_h) = v_{h3|\alpha\beta} - b^{\lambda}_{\alpha}b_{\lambda\beta}v_{h3} + b^{\lambda}_{\beta}\big|_{\alpha}v_{h\lambda} + b^{\lambda}_{\beta}v_{h\lambda|\alpha} + b^{\lambda}_{\alpha}v_{h\lambda|\beta} \qquad (5.2.84)$$

the substitution of equations (5.2.80) to (5.2.84) into equation (5.2.79) gives

$$\left.\begin{aligned}
&\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\big|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) \\
&\qquad - R^3_{\lambda 1}(\xi)a^{\nu\lambda}(\xi)[v_{h\nu|\varepsilon}(\xi)-b_{\varepsilon\nu}(\xi)v_{h3}(\xi)] \sum_{i=1}^{3} (\xi^\varepsilon_i-\xi^\varepsilon)(p^o_{j,i})_{,\alpha\beta} \\
&+ \{b^\gamma_\varepsilon(\xi)[-v_{h\gamma|\eta}(\xi)+b_{\gamma\eta}(\xi)v_{h3}(\xi)+\tfrac{1}{2}\,e_{\gamma\eta}(v_{h1,2}(\xi)-v_{h2,1}(\xi))]\} \times \\
&\qquad \times \{\sum_{i=1}^{3} (\xi^\varepsilon_i-\xi^\varepsilon)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}]\} \\
&+ \tfrac{1}{2}\,e_{\eta\gamma}a^{\gamma\lambda}(\xi)R^3_{\lambda 1}(\xi)(v_{h1,2}(\xi)-v_{h2,1}(\xi)) \times \\
&\qquad \times \sum_{i=1}^{3} [(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}] \\
&+ O(h)\sum_{k=1}^{3} [c^\ell_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k)+c^{\ell\lambda}_{j\alpha\beta}(v_{h\ell,\lambda}(\bar{\xi}_k)+c^{\varepsilon\eta}_{j\alpha\beta}v_{h3,\varepsilon\eta}(\bar{\xi}_k)]\ .
\end{aligned}\right\} \quad (5.2.85)$$

By using relation (5.2.68) and by observing that

$$\frac{1}{2}\,(v_{h1|2}(\xi)-v_{h2|1}(\xi))e_{\varepsilon\nu} + v_{h\nu|\varepsilon}(\xi) - b_{\varepsilon\nu}(\xi)v_{h3}(\xi) = \gamma_{\varepsilon\nu}(\vec{v}_h) \qquad (5.2.86)$$

and that, from relations (5.2.53) and (5.2.77),

$$R^3_{\lambda 1}(\xi) - (\xi^\varepsilon_i-\xi^\varepsilon)b_{\varepsilon\lambda}(\xi) = A^{\varepsilon\mu}_\lambda(\Sigma_i)b_{\varepsilon\mu}(\xi) \qquad (5.2.87)$$

we can rewrite relation (5.2.85) as follows :

$$\left.\begin{aligned}
&\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\big|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi) \times \\
&\qquad \times \sum_{i=1}^{3} A^{\varepsilon\mu}_\lambda(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\alpha\beta}] \\
&+ O(h)\sum_{i=1}^{3} [c^\ell_{j\alpha\beta}\,v_{h\ell}(\bar{\xi}_k)+c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\bar{\xi}_k)+c^{\varepsilon\eta}_{j\alpha\beta}v_{h3,\varepsilon\eta}(\bar{\xi}_k)]
\end{aligned}\right\} \quad (5.2.88)$$

that is exactly the expected result (5.2.37). Thus, theorem 5.2.2 is proved.

□

In order to obtain an estimate of type

$$\left|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)\right|_{0,K} \leq C h\{\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{2,K}\}^{1/2} , \quad (5.2.89)$$

it remains to estimate the second term of the second hand member of relation (5.2.88). Unfortunately, *one can check easily that, in general, this term is only in* O(1) *with respect to* h so that we cannot reach (5.2.89) directly. In order to circumvent this major difficulty, we have explored two ways :

a) *we have tried to prove that*

$$\left.\begin{aligned} & a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi) \times \\ & \quad \times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\alpha\beta}] \\ & = -\frac{1}{2}\{b^{\nu}_{\alpha}(\xi)\gamma_{\nu\beta}(\vec{v}_h) + b^{\nu}_{\beta}\gamma_{\nu\alpha}(\vec{v}_h)\} + O(h) \ . \end{aligned}\right\} \quad (5.2.90)$$

Such a relation should be of great interest since we could use in an equivalent way the *modified tensor of change of curvature* $\rho_{\alpha\beta}$ (see KOITER [1967]) :

$$\rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} - \frac{1}{2}(b^{\nu}_{\alpha}\gamma_{\nu\beta}+b^{\nu}_{\beta}\gamma_{\nu\alpha}) \quad (5.2.91)$$

so that relations (5.2.88) (5.2.90) (5.2.91) would give

$$\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\Big|_{K_j} - \rho_{\alpha\beta}(\vec{v}_h) = O(h) \ . \quad (5.2.92)$$

Unfortunately, we prove in Lemma 5.2.1 that relation (5.2.89) is not true in general.

b) *by introducing some restrictions* on the geometry of the middle surface, we show in Theorem 5.2.3 that the *second term of the right member of relation* (5.2.88) *is of order* O(ε), where the parameter ε measures the "shallowness" of the shell.

Lemma 5.2.1 (counterexample : case of a right circular cylindrical shell)
The relation (5.2.92) is not true in general.

Proof : Let the middle surface $\bar{\mathcal{S}}$ of the portion of a right circular cylinder be the image in $\mathcal{E}^3$ of a bounded subset $\bar{\Omega}$ of $\mathcal{E}^2$ (i.e. $\Omega =]-L,L[\times]-H,H[$, $0 < L \leq 2\pi R$, $0 < H$) by the regular mapping

$$\vec{\phi}(\xi^1,\xi^2) = R \cos \frac{\xi^1}{R} \vec{e}_1 + R \sin \frac{\xi^1}{R} \vec{e}_2 + \xi^2 \vec{e}_3 \quad . \tag{5.2.93}$$

According to relations (2.1.2) and (2.1.6), we obtain the covariant and contravariant basis as follows :

$$\vec{a}_1 = \vec{a}^1 = \begin{Bmatrix} -\sin \frac{\xi^1}{R} \\ \cos \frac{\xi^1}{R} \\ 0 \end{Bmatrix} \; ; \; \vec{a}_2 = \vec{a}^2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \; ; \; \vec{a}_3 = \vec{a}^3 = \begin{Bmatrix} \cos \frac{\xi^1}{R} \\ \sin \frac{\xi^1}{R} \\ 0 \end{Bmatrix} \tag{5.2.94}$$

The first and second fundamental forms are given by relations (2.1.3) (2.1.5) and (2.1.9), so that for this cylinder we get :

$$\left.\begin{array}{l} a_{11} = a^{11} = 1 \;\; ; \;\; a_{21} = a^{12} = 0 \;\; ; \;\; a_{22} = a^{22} = 1 \;\; ; \;\; a = 1 \\ b_{11} = b_1^1 = b^{11} = -\frac{1}{R} \text{ and all the other components} = 0 \; . \end{array}\right\} \tag{5.2.95}$$

Moreover the Christoffel symbols (see (2.1.14)) satisfy

$$\Gamma^{\lambda}_{\alpha\beta} = 0 \quad , \quad \alpha,\beta,\gamma = 1,2 \quad . \tag{5.2.96}$$

Now, let us particularize our counterexample in the following way : the triangulations $\mathcal{T}_h$ are constructed so that any triangle of $\mathcal{T}_h$ has one of its edges parallel to the cylinder axis, i.e., one of its edges is supported by a line ξ^1 = constant (for example $\xi^1_1 = \xi^1_3$: see Figure 5.2.1). This may be seen as an interesting manner to approximate the shell problem since each triangle $k = \vec{\phi}(K)$, $K \in \mathcal{T}_h$, has at least one of its edges lying in the middle surface $\bar{\mathcal{S}}$ of the cylinder.

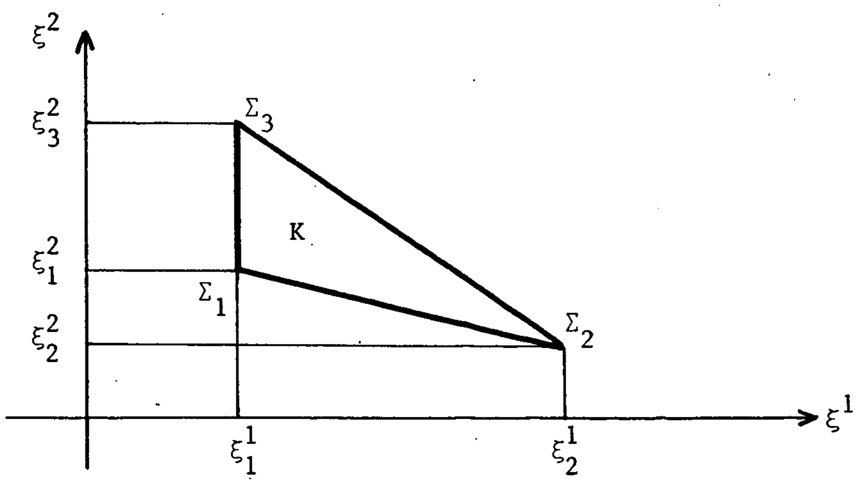


Figure 5.2.1 : one typical triangle $K \in \mathscr{T}_h$

$(h_1 = \xi^1_2 - \xi^1_1 \quad ; \quad h_2 = \xi^2_3 - \xi^2_1)$

Let us observe that for a general triangle of plane (ξ^1,ξ^2) the relation (5.2.26) implies

$$A^{\varepsilon\mu}_{\lambda}(\Sigma_i) = \frac{e_{\lambda\omega}}{2}\,\frac{(\xi^{\varepsilon}_{i+1}-\xi^{\varepsilon}_i)(\xi^{\mu}_{i+1}-\xi^{\mu}_i)(\xi^{\omega}_{i-1}-\xi^{\omega}_i) + (\xi^{\varepsilon}_{i-1}-\xi^{\varepsilon}_i)(\xi^{\mu}_{i-1}-\xi^{\mu}_i)(\xi^{\omega}_i-\xi^{\omega}_{i+1})}{(\xi^1_i-\xi^1_{i-1})(\xi^2_{i+1}-\xi^2_{i-1}) + (\xi^1_{i+1}-\xi^1_{i-1})(\xi^2_{i-1}-\xi^2_i)} \tag{5.2.97}$$

so that for a particular triangle as featured on Figure 5.2.1, we find

$$A^{11}_1(\Sigma_i) = \frac{1}{2}\,(\xi^1_1+\xi^1_2) - \xi^1_i \quad ; \quad A^{11}_2(\Sigma_i) = 0 \quad , \quad i=1,2,3 \ . \tag{5.2.98}$$

$$A^{12}_1(\Sigma_1) = \frac{1}{2}\,(\xi^2_2-\xi^2_1) \quad ; \quad A^{12}_1(\Sigma_2) = 0 \quad ; \quad A^{12}_1(\Sigma_3) = \frac{1}{2}\,(\xi^2_2-\xi^2_3) \tag{5.2.99}$$

$$A^{12}_2(\Sigma_1) = 0 \quad ; \quad A^{12}_2(\Sigma_2) = \frac{1}{2}\,(\xi^1_1-\xi^1_2) \quad ; \quad A^{12}_2(\Sigma_3) = 0 \ . \tag{5.2.100}$$

Then, by using relations (5.2.95) and $(5.2.98)_2$, we obtain

$$\left.\begin{aligned}
C_{\alpha\beta} &\overset{\text{def}}{=} a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi)\ \times \\
&\quad \times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_i)(p^1_{j,i,i-1})_{,\alpha\beta}] \\
&= -\frac{1}{R}\,\{\gamma_{11}(\vec{v}_h)\ \times \\
&\quad \times \sum_{i=1}^{3} A^{11}_1(\Sigma_i)[(\xi^1_{i+1}-\xi^1_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^1_{i-1}-\xi^1_i)(p^1_{j,i,i-1})_{,\alpha\beta}] \\
&\quad + \gamma_{12}(\vec{v}_h)\sum_{i=1}^{3} A^{11}_1(\Sigma_i)[(\xi^2_{i+1}-\xi^2_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^2_{i-1}-\xi^2_i)(p^1_{j,i,i-1})_{,\alpha\beta}]\}
\end{aligned}\right\} \tag{5.2.101}$$

or, by using relations (5.2.71) and $(5.2.98)_2$:

$$\left.\begin{aligned} C_{\alpha\beta} &= \frac{1}{R}\,\delta^1_\alpha\delta^1_\beta\,\gamma_{11}(\vec{v}_h) - \frac{1}{R}\,\gamma_{12}(\vec{v}_h)\,\times \\ &\times \sum_{i=1}^{3} A^{11}_1(\Sigma_i)[(\xi^2_{i+1}-\xi^2_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^2_{i-1}-\xi^2_i)(p^1_{j,i,i-1})_{,\alpha\beta}] \end{aligned}\right\}\quad (5.2.102)$$

On the other hand, relation (5.2.91) can be written :

$$\begin{aligned} \bar{\rho}_{\alpha\beta}(\vec{v}_h) &= \rho_{\alpha\beta}(\vec{v}_h) + \frac{1}{2}\,(b^\nu_\alpha\gamma_{\nu\beta}(\vec{v}_h)+b^\nu_\beta\gamma_{\nu\alpha}(\vec{v}_h)) \\ &= \rho_{\alpha\beta}(\vec{v}_h) + \frac{1}{2}\,(\delta^\varepsilon_\alpha\delta^\eta_\beta+\delta^\varepsilon_\beta\delta^\eta_\alpha)b^\nu_\varepsilon\gamma_{\nu\eta}(\vec{v}_h) \end{aligned}$$

or, with (5.2.95)

$$\bar{\rho}_{\alpha\beta}(\vec{v}_h) = \rho_{\alpha\beta}(\vec{v}_h) - \frac{1}{R}\,\delta^1_\alpha\delta^1_\beta\,\gamma_{11}(\vec{v}_h) - \frac{1}{2R}\,(\delta^1_\alpha\delta^2_\beta+\delta^2_\alpha\delta^1_\beta)\gamma_{12}(\vec{v}_h) \qquad (5.2.103)$$

Now, add relations (5.2.102) and (5.2.103), and use relation (5.2.71) :

$$\left.\begin{aligned} \bar{\rho}_{\alpha\beta}(\vec{v}_h) + C_{\alpha\beta} &= \rho_{\alpha\beta}(\vec{v}_h) + \frac{1}{R}\,\gamma_{12}(\vec{v}_h)\,\times \\ &\times \sum_{i=1}^{3} A^{12}_\lambda(\Sigma_i)\{(\xi^\lambda_{i+1}-\xi^\lambda_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^\lambda_{i-1}-\xi^\lambda_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} \\ - \frac{1}{R}\,\gamma_{12}(\vec{v}_h) &\sum_{i=1}^{3} A^{11}_1(\Sigma_i)\{(\xi^2_{i+1}-\xi^2_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^2_{i-1}-\xi^2_i)(p^1_{j,i,i-1})_{,\alpha\beta}\} \end{aligned}\right\}$$

Set

$$h_1 = \xi^1_2 - \xi^1_1 \quad\text{and}\quad h_2 = \xi^2_3 - \xi^2_1 \quad . \qquad (5.2.104)$$

Then, by using relations $(5.2.98)_1$ and (5.2.99) (5.2.100), we obtain

$$\bar{\rho}_{\alpha\beta}(\vec{v}_h) + C_{\alpha\beta} = \rho_{\alpha\beta}(\vec{v}_h) + \frac{h_1h_2}{2R}\,[(p^1_{j,3,1})_{,\alpha\beta}-(p^1_{j,1,3})_{,\alpha\beta}]\gamma_{12}(\vec{v}_h) \qquad (5.2.105)$$

so that relation (5.2.88) can be written

$$\left.\begin{aligned} \tilde{\rho}_{h\alpha\beta}(\tilde{\vec{v}}_h)\big|_{K_j} - \rho_{\alpha\beta}(\vec{v}_h) &= \frac{h_1h_2}{2R}\,[(p^1_{j,3,1})_{,\alpha\beta}-(p^1_{j,1,3})_{,\alpha\beta}]\gamma_{12}(\vec{v}_h) \\ + O(h)\sum_{i=1}^{3} &[c^\ell_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k)+c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\bar{\xi}_k)+c^{\varepsilon\eta}_{j\alpha\beta}v_{h3,\varepsilon\eta}(\bar{\xi}_k)] \;. \end{aligned}\right\}\quad (5.2.106)$$

For instance, consider the subtriangle K_1 (corresponding to j=1) and the values $\alpha=2$ and $\beta=2$ for the parameters α and β. From BERNADOU-BOISSERIE [2, page 79], we obtain

$$p^1_{1,3,1} = \frac{1}{4}(1+\eta_2)\lambda_1^3 - \frac{1}{4}(1+3\eta_2)\lambda_1^2\lambda_3 - \frac{1}{2}\lambda_1^2\lambda_2 + \lambda_1\lambda_3^2 + \lambda_1\lambda_2\lambda_3$$

$$p^1_{1,1,3} = -\frac{1}{4}(1+\eta_2)\lambda_1^3 + \frac{1}{4}(5+3\eta_2)\lambda_1^2\lambda_3 + \frac{1}{2}\lambda_1^2\lambda_2$$

so that

$$p \overset{\text{def}}{=} p^1_{1,3,1} - p^1_{1,1,3} = \frac{1}{2}(1+\eta_2)\lambda_1^3 - \frac{3}{2}(1+\eta_2)\lambda_1^2\lambda_3 - \lambda_1^2\lambda_2 + \lambda_1\lambda_3^2 + \lambda_1\lambda_2\lambda_3 \qquad (5.2.107)$$

where η_2 denotes the second eccentricity parameter. According to BERNADOU-BOISSERIE [2, (3.1.30)], we obtain (note that $\xi_1^1 = \xi_3^1$) :

$$p_{,22} = \frac{1}{(h_1h_2)^2}(h_1)^2\left\{\frac{\partial^2 p}{\partial\lambda_1^2} + \frac{\partial^2 p}{\partial\lambda_3^2} - 2\frac{\partial^2 p}{\partial\lambda_1\partial\lambda_3}\right\}$$

or

$$p_{,22} = (p^1_{1,3,1} - p^1_{1,1,3})_{,22} = \frac{1}{(h_2)^2}\{(11+9\eta_2)\lambda_1 - (7+3\eta_2)\lambda_3 - 4\lambda_2\} \ . \qquad (5.2.108)$$

Upon this expression, it is clear that $p_{,22} = O(h^{-2})$ with respect to the h-estimate, since the barycentric coordinates and the eccentricity parameters are undimensioned. So, by substituting into relation (5.2.106), we obtain

$$\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\big|_{K_j} - \rho_{\alpha\beta}(\vec{v}_h) = O(1) \qquad (5.2.109)$$

instead of the expected result (5.2.92). □

<u>Remark 5.2.2</u> : Of course, relations (5.2.91) and (5.2.109) involve immediately

$$\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\big|_{K_j} - \bar{\rho}_{\alpha\beta}(\vec{v}_h) = O(1) \ .$$

□

So, our first attempt to prove estimate (5.2.89) without any restriction on the geometry broke down. Now, *we are going to introduce some restrictions upon the geometry of the middle surface* which are consistent with KOITER's concept of "*quasi-shallow shells*".

<u>The case of "quasi-shallow" shells</u>

For any given bounded surface $\bar{\mathscr{S}}$ defined as in section 2.1, there exists two constants $\rho_1 > 0$ and $\rho_2 > 0$ such that (see BERNADOU-CIARLET [15, § 3.1]).

$$\rho_1 \geq \left\{ \begin{array}{l} |a_{\alpha\beta}|, |a^{\alpha\beta}|, |a|, |b_{\alpha\beta}|, |b^{\alpha}_{\beta}|, |b^{\alpha\beta}|, |a_{\alpha\beta,\lambda}|, |a^{\alpha\beta}{}_{,\lambda}|, |a_{,\lambda}| \\ |b_{\alpha\beta,\lambda}|, |b^{\alpha}_{\beta,\lambda}|, |b^{\alpha\beta}{}_{,\lambda}|, |\Gamma^{\alpha}_{\beta\lambda}|, |c_{\alpha\beta}| \end{array} \right\} \quad (5.2.110)$$

and

$$\rho_2 \leq a_{11}, a^{11}, a_{22}, a^{22}, a \ ; \quad (5.2.111)$$

in addition, there exists two constants $e_1 > 0$ and $e_2 > 0$ such that the thickness $e(\xi^1,\xi^2)$ satisfies

$$0 < e_1 \leq e(\xi^1,\xi^2) \leq e_2 \quad , \quad \forall (\xi^1,\xi^2) \in \bar{\Omega} \ . \quad (5.2.112)$$

Now, according to BERNADOU-ODEN [22, pp. 289-290], we shall be interested in the set of shells $\mathscr{S}_o(\rho_\alpha,e_\alpha,\omega_\alpha)$ and in its subset $\mathscr{S}_\varepsilon(\rho_\alpha,e_\alpha,\omega_\alpha)$ defined as follows.

<u>Definition 5.2.1</u> : The set $\mathscr{S}_o(\rho_\alpha,e_\alpha,\omega_\alpha)$. Given once and for all, seven constants ρ_1, ρ_2, e_1, e_2, ω_1, ω_2, R_o such that :

$$0 < \rho_2 \leq \rho_1 \quad , \quad 0 < e_1 \leq e_2 \quad , \quad 0 < \omega_1 \leq \omega_2 \quad , \quad R_o > 0 \quad , \quad (5.2.113)$$

we denote by $\mathscr{S}_o(\rho_\alpha,e_\alpha,\omega_\alpha)$ the set of shells $\bar{\mathscr{S}}$ such that :

(i) the inequalities (5.2.110) (5.2.111) and (5.2.112) are satisfied ;

(ii) the coefficients of the second fundamental form b_{α}^{β} are such that :

$$|b_{\alpha}^{\beta}| \ , \ |b_{\alpha|\lambda}^{\beta}| \leq \frac{1}{R_o} \ , \tag{5.2.114}$$

uniformly on $\bar{\mathcal{S}}$ for all α,β,λ ;

(iii) the reference domains Ω are sufficiently smooth, and, in particular :

$$C(\omega_1) \subset \Omega \subset C(\omega_2) \quad , \tag{5.2.115}$$

where the notation $C(\omega_2)$ denotes a square with a side of length ω_α.

□

Definition 5.2.2 : The subset $\mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$. Let there be given a parameter $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \frac{1}{R_o} \quad . \tag{5.2.116}$$

Then the set $\mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ is the set of all the shells $\mathcal{S} \in \mathcal{S}_o(\rho_\alpha, e_\alpha, \omega_\alpha)$ such that :

$$|b_{\alpha}^{\beta}| \ , \ |b_{\alpha|\lambda}^{\beta}| \leq \varepsilon \quad . \tag{5.2.117}$$

□

In particular, when the parameter ε is small with respect to ρ_1, the condition (5.2.117) implies that the normal curvatures and their variations are "small" : that means that the corresponding class $\mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ contains the shallow shells. Since we use the full linear expression for $\bar{\rho}_{\alpha\beta}$ (see (2.1.12)) instead of $u_{3|\alpha\beta}$, we will name this class, the set of *quasi-shallow shells*.

Now, let us state the following theorem :

<u>Theorem 5.2.3</u> : <u>There exists constants</u> $\varepsilon_o > 0$, $h_o > 0$, c_1 <u>and</u> c_2 <u>such that for any</u> $\varepsilon (0 < \varepsilon < \varepsilon_o)$, $h(0 < h < h_o)$, <u>the estimate</u> (5.2.37) <u>leads to</u>

$$|\tilde{\rho}_{h\beta}^{\alpha}(\tilde{v}_h)-\bar{\rho}_{\beta}^{\alpha}(\vec{v}_h)|_{0,K} \leq (c_1\varepsilon+c_2 h)\{\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{2,K}^2\}^{1/2} \quad (5.2.118)$$

<u>for any shell</u> $\mathcal{S} \in \mathcal{S}_{\varepsilon}(\rho_\alpha, e_\alpha, \omega_\alpha)$ <u>and for any</u> $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$ <u>and</u> $\vec{v}_h \in \vec{X}_h$ <u>associated through the bijection</u> F_h <u>defined in Theorem 4.2.1.</u>

<u>Proof</u> : The main difficulty is to estimate the second term of the right hand member of relation (5.2.37). Let us denote

$$\left.\begin{aligned} &\Delta_{\alpha\beta}(\vec{v}_h) = \\ &\quad \sum_{i=1}^{3} A_{\lambda}^{\varepsilon\mu}(\Sigma_i)[(\xi_{i+1}^{\eta}-\xi_i^{\eta})(p_{j,i,i+1}^1)_{,\alpha\beta}+(\xi_{i-1}^{\eta}-\xi_i^{\eta})(p_{j,i,i-1}^1)_{,\alpha\beta}] \times \\ &\quad \times b_{\varepsilon\mu}(\xi)\gamma_{\eta}^{\lambda}(\vec{v}_h) \quad . \end{aligned}\right\} \quad (5.2.119)$$

Then, from the relation (5.2.37), a proof similar to that of Theorem 5.2.1 gives the existence of a constant c_3, independent of h, such that

$$\left.\begin{aligned} &|\tilde{\rho}_{h\beta}^{\alpha}(\vec{\tilde{v}}_h)-\bar{\rho}_{\beta}^{\alpha}(\vec{v}_h)-a_h^{\alpha\nu}\Delta_{\nu\beta}(\vec{v}_h)|_{0,K} \leq \\ &\quad \leq c_3 h\{\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{2,K}^2\}^{1/2} \ . \end{aligned}\right\} \quad (5.2.120)$$

In addition, since the sets $\mathcal{S}_{\varepsilon}(\rho_\alpha, e_\alpha, \omega_\alpha)$ are subsets of the set $\mathcal{S}_o(\rho_\alpha, e_\alpha, \omega_\alpha)$ which is defined through the compact sets $[0,\rho_1]$, $[\rho_2,\rho_1]$, $[e_1,e_2]$ (see (5.2.110) to (5.2.112)), a proof by contradiction reveals immediately that the constants c_3 can be chosen independent of ε.

Now let us prove that there exists a constant $c_{\alpha\beta}$, independent of h and ε, such that

$$|\Delta_{\alpha\beta}(\vec{v}_h)|_{0,K} \leq c_{\alpha\beta}\ \varepsilon\{\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{0,K}^2\}^{1/2} \ . \quad (5.2.121)$$

Firstly from the expressions (5.2.25)(5.2.26) we have

$$A_{\lambda}^{\varepsilon\mu}(\Sigma_i) = O(h) \ .$$

Next, a proof similar to that of relation (5.2.108) gives

$$(p_{j,i,i+1}^{1})_{,\alpha\beta} = O(h^{-2}) \ .$$

Thus, the coefficient

$$\sum_{i=1}^{3} A_{\lambda}^{\varepsilon\mu}(\Sigma_i)[(\xi_{i+1}^{\eta}-\xi_i^{\eta})(p_{j,i,i+1}^{1})_{,\alpha\beta} + (\xi_{i-1}^{\eta}-\xi_i^{\eta})(p_{j,i,i-1}^{1})_{,\alpha\beta}]$$

is

i) just defined upon the triangulation of the plane reference domain Ω ; in particular, this term is completely independent of the middle surface and thus, it is independent of ε ;

ii) of order O(1) with respect to h.

On the other hand, from inequalities (5.2.110) (5.2.111) and (5.2.117), we can prove that

$$|b_{\varepsilon\mu}(\xi)\gamma_{\eta}^{\lambda}(\vec{v}_h)|_{0,K} \leq C\varepsilon\{\|v_{h1}\|_{1,K}^2 + \|v_{h2}\|_{1,K}^2 + \|v_{h3}\|_{0,K}^2\}^{1/2} \tag{5.2.122}$$

where the constant C is independent of h and ε. Then, the estimate (5.2.121) is proved.

Finally from estimates (5.2.110) (5.2.111) (5.2.120) and (5.2.121) we easily derive the expected estimate (5.2.118).

□

Of course, we would wished firstly to prove that the estimate (5.2.118) was valid with $C_1 \equiv 0$, i.e., without any restrictions on the geometry of the middle surface. In fact such a result does not seem to be true :

i) with respect to the mathematical analysis, we refer to lemma 5.2.1 ;

ii) with respect to numerical experiments and engineering experience : for instance ZIENKIEWICZ [23, p. 329], IRONS and AHMAD [24, p. 281] and GALLAGHER [25, p. 5] pointed out that some extraneous bending effects may occur, especially in regions where the membrane state of stress predominates (this is in conformity with our result (5.2.88)).

So, *subsequently, we will complete the proof of convergence in the case of quasi-shallow shells within the validity of Theorem 5.2.3.*

First, we are able to prove the following theorem giving the estimate of the consistency term $|a(\vec{v}_h,\vec{w}_h) - a_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)|$.

Theorem 5.2.4 : There exists constants $\varepsilon_o > 0$, $h_o > 0$, $c > 0$ such that for any $\varepsilon(0<\varepsilon<\varepsilon_o)$, for any $h(0<h<h_o)$, for any shell $\mathcal{S} \in \mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha,\omega_\alpha)$ and for any $\tilde{\vec{v}}_h,\tilde{\vec{w}}_h \in \tilde{\vec{X}}_h$ and $\vec{v}_h,\vec{w}_h \in \vec{X}_h$ in correspondence through the bijection F_h defined in Theorem 4.2.1, we have

$$|a(\vec{v}_h,\vec{w}_h)-\tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \le C(\varepsilon+h)\|\vec{v}_h\|\|\vec{w}_h\| \quad . \tag{5.2.123}$$

Proof : By virtue of (5.2.4), it suffices to estimate $|ER_{K1}(\vec{v}_h,\vec{w}_h)|$, the analysis being the same for the other terms $ER_{Ki}(\vec{v}_h,\vec{w}_h)$, $2\le i\le 4$. Applying relation (5.2.27), we obtain

$$\left|\sqrt{a} - \sqrt{a_h}\right|_{0,\infty,K} \le Ch \quad . \tag{5.2.124}$$

Since the mappings $a_{\alpha\beta}$, $a^{\alpha\beta}$, a, $b_{\alpha\beta}$, b^α_β are independent of h, we deduce from (5.2.9) (5.2.13) and (5.2.124) :

$$|ER_{K1}(\vec{v}_h,\vec{w}_h)| \le Ch \|\vec{v}_h\|_K \|\vec{w}_h\|_K \quad ,$$

and, in a similar way

$$|ER_{K2}(\vec{v}_h,\vec{w}_h)| \leq Ch \, \|\vec{v}_h\|_K \, \|\vec{w}_h\|_K \, .$$

From an analogous decomposition to (5.2.9) and from (5.2.118) and (5.2.124), we can prove that

$$|ER_{Ki}(\vec{v}_h,\vec{w}_h)| \leq C(h+\varepsilon) \, \|\vec{v}_h\|_K \, \|\vec{w}_h\|_K \quad , \; i = 3, 4 \, .$$

Then, the proof of estimate (5.2.123) is achieved by summation over all the triangles K of $\mathcal{T}_h$.

$\square$

Remark 5.2.3 : Order of magnitude of different terms in (5.2.37) :

KOITER [26, pp. 16-17 and 30-31] proposes two different expressions to get an approximate expression of the strain energy per unit area of the undeformed middle surface :

$$\bar{V} = \frac{1}{2} eE^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{1}{24} e^3 E^{\alpha\beta\lambda\mu} \bar{\rho}_{\alpha\beta}\bar{\rho}_{\lambda\mu}$$

and

$$V = \frac{1}{2} eE^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{1}{24} e^3 E^{\alpha\beta\lambda\mu} \rho_{\alpha\beta}\rho_{\lambda\mu}$$

where the expressions of $\bar{\rho}_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are linked by the relation

$$\bar{\rho}_{\alpha\beta} = \rho_{\alpha\beta} + \frac{1}{2} b_\alpha^\lambda \gamma_{\lambda\beta} + \frac{1}{2} b_\beta^\lambda \gamma_{\lambda\alpha} \quad .$$

From these relations, we obtain

$$\bar{V} - V = \frac{e^3}{24} E^{\alpha\beta\lambda\mu} \{2b_\alpha^\kappa \gamma_{\kappa\beta}\rho_{\lambda\mu} + b_\alpha^\kappa b_\lambda^\omega \gamma_{\kappa\beta}\gamma_{\omega\mu}\} \, .$$

In order to state the "equivalence" between V and $\bar{V}$, one can use the arguments of KOITER [26, part III] which allow to find simplified equations. They are based on the respective orders of magnitude of the characteristic terms as

$\frac{1}{R}$ (absolute) maximum of the curvatures $|b_\alpha^\beta|$

γ (absolute) maximum of strain tensor $|\gamma_{\alpha\beta}|$

ρ (absolute) maximum of change of curvature tensor $|\rho_{\alpha\beta}|$

In this way, within the framework of a *thin shell theory with small deformations*, KOITER [1966, pp. 35-38] considers three cases which correspond to the position of $\frac{e\rho}{\gamma}$ with respect to the poles $\frac{e}{R}$ and $\mathrm{Min}(\frac{R}{e},\frac{1}{e\rho})$. These quantities satisfy $\frac{e}{R} \ll \mathrm{Min}(\frac{R}{e},\frac{1}{e\rho})$.

<u>Case 1</u> : $\frac{e}{R} \ll \frac{e\rho}{\gamma} \ll \mathrm{Min}(\frac{R}{e},\frac{1}{e\rho})$

In this range, flexural and extensional strains are of comparable orders of magnitude. That means in particular that $\frac{\gamma}{R} \ll \rho$ and then the difference between $\bar{\rho}_{\alpha\beta}$ and $\rho_{\alpha\beta}$ is neglectible.

<u>Case 2</u> : $O[\mathrm{Min}(\frac{R}{e},\frac{1}{e\rho})] \leq \frac{e\rho}{\gamma}$

In this range, flexural strains are greater than extensional strains. Since $\frac{e}{R} \ll \mathrm{Min}(\frac{R}{e},\frac{1}{e\rho})$, we obtain

$$\frac{e}{R} \ll \frac{e\rho}{\gamma} \quad \text{or} \quad \frac{\gamma}{R} \ll \rho \quad ,$$

and we can conclude like in case 1.

<u>Case 3</u> : $\frac{e\rho}{\gamma} \ll O(\frac{e}{R})$

In this range, extensional strains are greater than flexural strains. In this case $\bar{\rho}_{\alpha\beta} - \rho_{\alpha\beta}$ is not neglectible so that $\bar{V}$ and V are not equivalent.

Now let us examine the estimate (5.2.37). In the right hand member, the second term is in $O(\frac{\gamma}{R})$. That means

i) <u>in Case 1 or Case 2</u> (flexural strains are preponderant) : we can drop the second term of the right hand member without changing significantly the value of the strain energy. Although we have not realized a precise study of the corresponding approximation, we can consider that in these cases

$$\tilde{\rho}_{h\alpha\beta}(\vec{\tilde{v}}_h)\Big|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + O(h)$$

and then we guess that convergence is valid without any restrictions on the geometry of the middle surface ;

ii) <u>in Case 3</u> (extensional strains are preponderant) : as mentionned before, the second term of the right hand member of (5.2.37) is no longer neglectible with respect to $\bar{\rho}_{\alpha\beta}(\vec{v}_h)$. This fact is in particular corroborated by the example reported by GALLAGHER [25, p.5] who considers a counterexample where the extensional strains are preponderant.

□

In order to estimate $|\tilde{a}_h(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) - a_h^*(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)|$ which appears into relation (5.2.1), let us prove the following theorem : it gives a relation between $\|.\|$ and $\|.\|_h$ where the notation $\|.\|_h$ is introduced in (4.2.39).

<u>Theorem 5.2.5</u> : <u>For any functions</u> $\vec{\tilde{v}}_h \in \vec{\tilde{X}}_h$ <u>and</u> $\vec{v}_h \in \vec{X}_h$ <u>linked by the bijection</u> F_h <u>introduced in theorem 4.2.1, there exists a constant</u> c, <u>independent of</u> h, <u>such that</u>

$$\|\vec{\tilde{v}}_h\|_h \leq c\,\|\vec{v}_h\| \quad . \tag{5.2.125}$$

<u>Proof</u> :

<u>Step 1</u> : $\displaystyle\sum_{K\in\mathscr{C}_h} \|\tilde{v}_{h\beta}-v_{h\beta}\|_{1,K} \leq c\,\|\vec{v}_h\|$

From (5.2.14), (5.2.16), (5.2.28), we find

$$\tilde{v}_{h\beta}(\xi) = \sum_{i=1}^{3} \lambda_i\ \{(\delta_\beta^\nu + O(h)c_\beta^\nu)v_{h\nu}(\Sigma_i) + O(h)c_\beta v_{h3}(\Sigma_i)\}$$

$$= v_{h\beta}(\xi) + O(h)c_\beta \sum_{i=1}^{3}\sum_{j=1}^{3} v_{hj}(\Sigma_i)$$

so that

$$|\tilde{v}_{h\beta} - v_{h\beta}|_{0,K} \leq c\ h^2 \sum_{j=1}^{3} |v_{hj}|_{0,\infty,K}\ .$$

Then, some results of interpolation theory in Sobolev spaces show that (CIARLET [5, Theorem 3.1.2]) :

$$|v_{hj}|_{0,\infty,K} \leq c|\hat{v}_{hj}|_{0,\infty,\hat{K}} \leq c|\hat{v}_{hj}|_{0,\hat{K}} \leq c\ h^{-1}\ |v_{hj}|_{0,K}\ .$$

Finally we obtain

$$|\tilde{v}_{h\beta} - v_{h\beta}|_{0,K} \leq c\ h \sum_{j=1}^{3} |v_{hj}|_{0,K}$$

hence

$$||\tilde{v}_{h\beta} - v_{h\beta}||_{0,\Omega} \leq c\ h\ ||\vec{v}_h||_{0,\Omega}\ . \qquad (5.2.126)$$

Next, using the arguments of Step 3 and 4 of the proof of Theorem 5.2.1, we obtain :

$$||\tilde{v}_{h\beta,\nu} - v_{h\beta,\nu}||_{0,K} \leq c\ ||\vec{v}_h||_{0,K} + c\ h\ |\vec{v}_h|_{1,K}$$

so that

$$\sum_{K\in\mathcal{T}_h} ||\tilde{v}_{h\beta,\nu} - v_{h\beta,\nu}||_{0,K} \leq c\ ||\vec{v}_h||_{1,\Omega} \qquad (5.2.127)$$

Then, considering (5.2.126) (5.2.127) we have

$$\sum_{K\in\mathcal{T}_h} ||\tilde{v}_{h\beta} - v_{h\beta}||_{1,K} \leq c\ ||\vec{v}_h||\ . \qquad (5.2.128)$$

<u>Step 2</u> : $\sum_{K\in\mathcal{T}_h} ||\tilde{v}_{h3} - v_{h3}||_{2,K}$

First, from (5.2.64) we have

$$\tilde{v}_{h3}(\Sigma_i) = v_{h3}(\Sigma_i) + O(h) \sum_{j=1}^{3} v_{hj}(\Sigma_i) \ . \tag{5.2.129}$$

Next, from (5.2.65) we derive

$$\left.\begin{aligned} \tilde{v}_{h3,\nu}(\Sigma_i) &= v_{h3,\nu}(\Sigma_i) + b_\nu^\lambda(\Sigma_i)v_{h\lambda}(\Sigma_i) \\ &\quad + O(h)\{c_{3\lambda}v_{h3,\lambda}(\Sigma_i) + c_\varepsilon\, v_{h\varepsilon}(\Sigma_i)\}. \end{aligned}\right\} \tag{5.2.130}$$

But, by definition of space X_{h2}, we have on any subtriangle K_j, j=1,2,3, of the given triangle K with vertices $\Sigma_i = (\xi_i^1, \xi_i^2)$, i=1,2,3,

$$\tilde{v}_{h3}(\xi)\big|_{K_j} = \sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i)p_{j,i}^{o}(\lambda) + \sum_{i=1}^{3} \{(\xi_{i+1}^\nu - \xi_i^\nu)p_{j,i,i+1}^{1}(\lambda)$$

$$+ (\xi_{i-1}^\nu - \xi_i^\nu)p_{j,i,i-1}^{1}(\lambda)\}\tilde{v}_{h3,\nu}(\Sigma_i)$$

where $\tilde{v}_{h3|K_j}$ denotes the restriction of function $\tilde{v}_{h3}$ to subtriangle K_j, and we have a similar expression for $v_{h3}(\xi)\,|_{K_j}$.

Then, we deduce for any subtriangle K_j , j=1,2,3, and any $\xi \in K_j$

$$\tilde{v}_{h3}(\xi)\big|_{K_j} = v_{h3}(\xi)\big|_{K_j} + O(h)\{ \sum_{i=1}^{3} \sum_{j=1}^{3} c_j v_{hj}(\Sigma_i) + \sum_{i=1}^{3} c_{3\lambda}\, v_{h3,\lambda}(\Sigma_i)\} \ ,$$

and thus

$$|\tilde{v}_{h3} - v_{h3}|_{0,K} \le c\ h^2 \ \{ \sum_{j=1}^{3} |v_{hj}|_{0,\infty,K} + |v_{h3}|_{1,\infty,K}\} \ .$$

Results of interpolation theory in Sobolev spaces give

$$|\tilde{v}_{h3} - v_{h3}|_{0,K} \le c\ h\ \{ \sum_{j=1}^{3} |v_{hj}|_{0,K} + |v_{h3}|_{1,K}\}$$

and thus

$$||\tilde{v}_{h3} - v_{h3}||_{0,\Omega} \leq c\, h\, ||\vec{v}_h|| \, . \qquad (5.2.131)$$

In the same way, since, for any subtriangle K_j, $j=1,2,3$, and any $\xi \in K_j$, we have

$$\tilde{v}_{h3,\alpha}(\xi)\big|_{K_j} = \sum_{i=1}^{3} \tilde{v}_{h3}(\Sigma_i)(p^o_{j,i})_{,\alpha} + \sum_{i=1}^{3} \{(\xi^\nu_{i+1}-\xi^\nu_i)(p^1_{j,i,i+1})_{,\alpha}$$

$$+ (\xi^\nu_{i-1}-\xi^\nu_i)(p^1_{j,i,i-1})_{,\alpha}\}\tilde{v}_{h3,\nu}(\Sigma_i)$$

and a similar expression for $v_{h3,\alpha}(\xi)|_{K_j}$, we deduce from (5.2.129) and (5.2.130)

$$\tilde{v}_{h3,\alpha}(\xi)\big|_{K_j} = v_{h3,\alpha}(\xi)\big|_{K_j} + O(1)\, \{\sum_{i=1}^{3}\sum_{j=1}^{3} c_j\, v_{hj}(\Sigma_i) + \sum_{i=1}^{3} c_{3\lambda}\, v_{h3,\lambda}(\Sigma_i)\} \, ,$$

and thus

$$|\tilde{v}_{h3}-v_{h3}|_{1,K} \leq c\, \{\sum_{j=1}^{3} |v_{hj}|_{0,K} + |v_{h3}|_{1,K}\} \, .$$

Hence, by combining this last inequality with (5.2.131) , we obtain

$$\sum_{K\in\mathcal{T}_h} ||\tilde{v}_{h3}-v_{h3}||_{1,K} \leq c||\vec{v}_h|| \, . \qquad (5.2.132)$$

Finally, finite expansion (5.2.79) gives

$$\tilde{v}_{h3,\alpha\beta}(\xi)\big|_{K_j} = v_{h3,\alpha\beta}(\xi)\big|_{K_j} + O(1)\, \{\sum_{j=1}^{3} c_j v_{hj}(\xi) + \sum_{j=1}^{3} c_{j\varepsilon} v_{hj,\varepsilon}(\xi)\}$$

so that

$$|\tilde{v}_{h3} - v_{h3}|_{2,K} \leq c \sum_{j=1}^{3} \{|v_{hj}|_{0,K} + |v_{hj}|_{1,K}\}$$

and thus, by combining with (5.2.132)

$$\sum_{K\in\mathcal{T}_h} ||\tilde{v}_{h3}-v_{h3}||_{2,K} \leq c||\vec{v}_h|| \, . \qquad (5.2.133)$$

Inequality (5.2.125) immediatly follows from (5.2.128) , (5.2.133) and triangular inequality.

□

Now, to achieve the evaluation of the consistency error, it remains to consider the difference

$$|\tilde{a}_h(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) - a_h^*(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)| \ .$$

From definitions (4.3.3) and (4.3.14) we have for any $\vec{\tilde{v}}_h,\vec{\tilde{w}}_h \in \vec{\tilde{X}}_h$

$$\left.\begin{aligned}
\tilde{a}_h(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) = \sum_{K\in\mathcal{T}_h} \int_K \frac{Ee}{1-\nu^2} \{ & (1-\nu)\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)\tilde{\gamma}^{\beta}_{h\alpha}(\vec{\tilde{w}}_h) \\
& + \nu\, \tilde{\gamma}^{\alpha}_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\gamma}^{\beta}_{h\beta}(\vec{\tilde{w}}_h) \\
& + \frac{e^2}{12} [(1-\nu)\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)\tilde{\rho}^{\beta}_{h\alpha}(\vec{\tilde{w}}_h)+\nu\tilde{\rho}^{\alpha}_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\rho}^{\beta}_{h\beta}(\vec{\tilde{w}}_h)]\}\sqrt{a_h}\, d\xi^1 d\xi^2
\end{aligned}\right\} ,$$

and

$$\left.\begin{aligned}
a_h^*(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) = \sum_{K\in\mathcal{T}_h} \sqrt{a_h} \Big[\sum_{\ell_1=1}^{L_1} & \omega_{\ell_{1,K}} \Big(\frac{Ee}{1-\nu^2}\{(1-\nu)\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)\tilde{\gamma}^{\beta}_{h\alpha}(\vec{\tilde{w}}_h) \\
& + \nu\tilde{\gamma}^{\alpha}_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\gamma}^{\beta}_{h\beta}(\vec{\tilde{w}}_h)\}\Big)(b_{\ell_{1,K}}) \\
& + \sum_{i=1}^{3}\sum_{\ell_2=1}^{L_2} \omega_{\ell_{2,K_i}} \Big(\frac{Ee^3}{12(1-\nu^2)}\{(1-\nu)\tilde{\rho}^{\alpha}_{h\beta}(\vec{\tilde{v}}_h)\tilde{\rho}^{\beta}_{h\alpha}(\vec{\tilde{w}}_h) \\
& + \nu\tilde{\rho}^{\alpha}_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\rho}^{\beta}_{h\beta}(\vec{\tilde{w}}_h)\}\Big)(b_{\ell_{2,K_i}})\Big] \ .
\end{aligned}\right\}$$

Using the definition (4.3.12) we obtain

$$\tilde{a}_h(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)-a_h^*(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) = \sum_{K\in\mathcal{T}_h} \sqrt{a_h}\{E_{K_M}(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)+\sum_{i=1}^{3} E_{K_{iB}}(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)\} \tag{5.2.134}$$

where

$$\left.\begin{aligned} E_{K_M}(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) &= \int_K [\frac{Ee}{1-\nu^2}\{(1-\nu)\tilde{\gamma}^\alpha_{h\beta}(\vec{\tilde{v}}_h)\tilde{\gamma}^\beta_{h\alpha}(\vec{\tilde{w}}_h) \\ &\quad + \nu\tilde{\gamma}^\alpha_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\gamma}^\beta_{h\beta}(\vec{\tilde{w}}_h)\}]d\xi^1 d\xi^2 \\ &\quad - \sum_{\ell_1=1}^{L_1} \omega_{\ell_{1,K}} (\frac{Ee}{1-\nu^2}\{(1-\nu)\tilde{\gamma}^\alpha_{h\beta}(\vec{\tilde{v}}_h)\tilde{\gamma}^\beta_{h\alpha}(\vec{\tilde{w}}_h) \\ &\quad + \nu\tilde{\gamma}^\alpha_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\gamma}^\beta_{h\beta}(\vec{\tilde{w}}_h)\})(b_{\ell_1,K}) \end{aligned}\right\} \tag{5.2.135}$$

and

$$\left.\begin{aligned} E_{K_{iB}}(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h) &= \int_{K_i} \frac{Ee^3}{12(1-\nu^2)}\{(1-\nu)\tilde{\rho}^\alpha_{h\beta}(\vec{\tilde{v}}_h)\tilde{\rho}^\beta_{h\alpha}(\vec{\tilde{w}}_h) \\ &\quad + \nu\tilde{\rho}^\alpha_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\rho}^\beta_{h\beta}(\vec{\tilde{w}}_h)\}d\xi^1 d\xi^2 \\ &\quad - \sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i}[\frac{Ee^3}{12(1-\nu^2)}\{(1-\nu)\tilde{\rho}^\alpha_{h\beta}(\vec{\tilde{v}}_h)\tilde{\rho}^\beta_{h\alpha}(\vec{\tilde{w}}_h) \\ &\quad + \nu\tilde{\rho}^\alpha_{h\alpha}(\vec{\tilde{v}}_h)\tilde{\rho}^\beta_{h\beta}(\vec{\tilde{w}}_h)\}](b_{\ell_2,K_i}) \end{aligned}\right\} \tag{5.2.136}$$

The purpose of the next two theorems is to estimate the errors (5.2.135) and (5.2.136).

<u>Theorem 5.2.6</u> : <u>With the notations of section 4.3, we assume that</u>

$$\forall\ \hat{\phi} \in P_0(\hat{K}) \quad , \quad \hat{E}_1(\hat{\phi}) = 0 \tag{5.2.137}$$

<u>where</u>

$$\hat{E}_1(\hat{\phi}) = \int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x} - \sum_{\ell_1=1}^{L_1} \hat{\omega}_{\ell_1}\ \hat{\phi}(\hat{b}_{\ell_1})\ . \tag{5.2.138}$$

Then there exists a constant c independent of $K \in \mathcal{T}_h$ and h such that

$$|E_{K_M}(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)| \leq c\, h_K \left(\sum_{\alpha=1}^{2} ||\tilde{v}_{h\alpha}||_{1,K}^2\right)^{1/2} \left(\sum_{\alpha=1}^{2} ||\tilde{w}_{h\alpha}||_{1,K}^2\right)^{1/2} . \qquad (5.2.139)$$

Proof : For simplicity we have denoted E_{K_M} as a function of $\vec{\tilde{v}}_h, \vec{\tilde{w}}_h$. More precisely E_{K_M} is just a function of the two first components $\tilde{v}_{h\alpha}$ and $\tilde{w}_{h\alpha}$. The corresponding function to integrate in (5.2.135) is the product of functions $Ee/1-\nu^2$, $(1-\nu), \nu$, $a_h^{\alpha\beta}$ by functions $\tilde{\gamma}_{h\alpha\beta}(\vec{\tilde{v}}_h) = \frac{1}{2}(\tilde{v}_{h\alpha,\beta}+\tilde{v}_{h\beta,\alpha})$.

By definition the functions $Ee/1-\nu^2$, $1-\nu$, ν are uniformly bounded in $\bar{\Omega}$. From (5.2.27) we can check that

$$a_h^{\alpha\beta} = a^{\alpha\beta}(\xi) + 0(h)$$

so that for h sufficiently small the function $a_h^{\alpha\beta}$ is uniformly bounded in $\bar{\Omega}$, independently of h. Then the theorem 5.2.6 is an immediate consequence of CIARLET [5, Theorem 4.1.4].

Theorem 5.2.7 : Assume that

$$\forall\, \hat{\phi} \in P_2(\hat{K}) \quad , \quad \hat{E}_2(\hat{\phi}) = 0 \qquad (5.2.140)$$

where

$$\hat{E}_2(\hat{\phi}) = \int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x} - \sum_{\ell_2=1}^{L_2} \hat{\omega}_{\ell_2}\, \hat{\phi}(\hat{b}_{\ell_2}) . \qquad (5.2.141)$$

Then there exists a constant c independent of $K \in \mathcal{T}_h$ and h such that

$$|E_{K_{iB}}(\vec{\tilde{v}}_h,\vec{\tilde{w}}_h)| \leq c\, h_K ||\tilde{v}_{h3}||_{2,K_i} ||\tilde{w}_{h3}||_{2,K_i} . \qquad (5.2.142)$$

Proof : Observe that $E_{K_{iB}}$ is only function of $\tilde{v}_{h3}$ and $\tilde{w}_{h3}$ as a consequence of the expressions (5.2.136) and (4.3.2), i.e. $\tilde{\rho}_{h\alpha\beta} = \tilde{v}_{h3,\alpha\beta}$. Then theorem 5.2.7 can be proved similarly to theorem 5.2.6. □

Remark 5.2.4 : Analogously to BERNADOU-DUCATEL [8, Theorem 4.3] we could prove that $|E_{K_{iB}}(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \leq c\ h_K\ ||\tilde{v}_{h3}||_{3,K_i}||\tilde{w}_{h3}||_{2,K_i}$ only assuming that $\forall\ \hat{\phi} \in P_1(\hat{K})\ ,\ \hat{E}_2(\hat{\phi}) = 0$. But subsequently we need (5.2.142) to derive the property (5.1.1). □

Remark 5.2.5 : Let us emphasize that the bilinear form $a_h^*(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)$ is evaluated subtriangle by subtriangle K_i. Particularly in Theorem 5.2.7 the reference triangle $\hat{K}$ is successively in correspondence with every subtriangle K_i. It is a consequence of the definition (4.2.4) of $\tilde{v}_{h3},\tilde{w}_{h3}$ as polynomials on each subtriangle K_i but not on each triangle K. □

Combining the results of theorems 5.2.5, 5.2.6, 5.2.7, we finally find :

Theorem 5.2.8 : Assume that

$$\forall\ \hat{\phi} \in P_0(\hat{K})\quad ,\quad \hat{E}_1(\hat{\phi}) = 0 \quad \text{(for membrane terms)} \tag{5.2.143}$$

$$\forall\ \hat{\phi} \in P_2(\hat{K})\quad ,\quad \hat{E}_2(\hat{\phi}) = 0\ , \quad \text{(for bending terms)} \tag{5.2.144}$$

with $\hat{E}_1$ and $\hat{E}_2$ defined by (5.2.138) and (5.2.141). Then, there exists constants $\varepsilon_o > 0$, $h_o > 0$, $c > 0$ such that for any $\varepsilon(0<\varepsilon<\varepsilon_o)$, for any $h\ (0<h<h_o)$, for any shell $\mathcal{S} \in \mathcal{S}_\varepsilon(\rho_\alpha,e_\alpha,w_\alpha)$, we have

$$|a(\vec{v}_h,\vec{w}_h) - b_h(\vec{v}_h,\vec{w}_h)| \leq c(h+\varepsilon)||\vec{v}_h||\ ||\vec{w}_h||\ , \tag{5.2.145}$$

for any $\vec{v}_h \in \vec{X}_h$ and for any $\vec{w}_h \in \vec{X}_h$.

Proof : (i) From Theorem 5.2.4 we have

$$|a(\vec{v}_h,\vec{w}_h) - \tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \le c(h+\varepsilon)||\vec{v}_h||\ ||\vec{w}_h||. \qquad (5.2.146)$$

(ii) Using theorems 5.2.6 and 5.2.7, finite expansion (5.2.60) and relation (5.2.134), we find for h sufficiently small

$$|\tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h) - a^*_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \le c\ h\ ||\tilde{\vec{v}}_h||_h\ ||\tilde{\vec{w}}_h||_h\ ,$$

where the notation $||.||_h$ was introduced in (4.2.39). From theorem 5.2.5, we have $||\tilde{\vec{v}}_h||_h \le c\ ||\vec{v}_h||$ and $||\tilde{\vec{w}}_h||_h \le c\ ||\vec{w}_h||$, so that

$$|\tilde{a}_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h) - a^*_h(\tilde{\vec{v}}_h,\tilde{\vec{w}}_h)| \le c\ h\ ||\vec{v}_h||\ ||\vec{w}_h||\ . \qquad (5.2.147)$$

(iii) Then, the estimate (5.2.145) arises from (5.2.146) (5.2.147) and (4.3.16). □

5.3 - Estimate of consistency error $|f(\vec{w}_h)-g_h(\vec{w}_h)|$.

Similarly to theorem 5.2.8, we obtain the following result :

Theorem 5.3.1 : Assume that $\hat{E}_1$ and $\hat{E}_2$ respectively defined in (5.2.138) and (5.2.141) satisfy the properties (5.2.143) and (5.2.144).

Then, for any regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ and for any components $p^i \in W^{1,q}(\Omega)$, $q \in \mathbb{R}$, $q > 2$, defined in (2.2.4), there exists a constant c, independent of h, such that, for h sufficiently small and for any $\vec{w}_h \in \vec{V}_h$,

$$|f(\vec{w}_h) - g_h(\vec{w}_h)| \le c\ h\ ||\vec{p}||_{1,q,\Omega}\ ||\vec{w}_h||\ . \qquad (5.3.1)$$

Proof : (in three steps)

Step 1 : Let us consider any function $\vec{w}_h \in \vec{X}_h$ and let us denote

$\tilde{\vec{w}}_h \in \tilde{\vec{X}}_h$ the corresponding function through the bijection F_h defined in Theorem 4.2.1. From the definitions (4.3.4)(4.3.15)(4.3.17), we obtain

$$f(\vec{w}_h) - g_h(\vec{w}_h) = f(\vec{w}_h) - \tilde{f}_h(\tilde{\vec{w}}_h) + \tilde{f}_h(\tilde{\vec{w}}_h) - f_h^*(\tilde{\vec{w}}_h)$$

and thus, to evaluate the consistency error, we have to estimate the two terms $|f(\vec{w}_h)-\tilde{f}_h(\tilde{\vec{w}}_h)|$ and $|\tilde{f}_h(\tilde{\vec{w}}_h)-f_h^*(\tilde{\vec{w}}_h)|$.

<u>Step 2</u> : <u>Estimate of</u> $|f(\vec{w}_h)-\tilde{f}_h(\tilde{\vec{w}}_h)|$.

From definition (4.3.4), we obtain

$$|f(\vec{w}_h)-\tilde{f}_h(\tilde{\vec{w}}_h)| \leq |\int_\Omega \vec{p}.(\vec{w}_h-\tilde{\vec{w}}_h)\sqrt{a}\, d\xi^1 d\xi^2| + |\int_\Omega \vec{p}.\tilde{\vec{w}}_h(\sqrt{a_h}-\sqrt{a})d\xi^1 d\xi^2| .$$

By using (5.2.126) and (5.2.131), we deduce for the first integral

$$|\int_\Omega \vec{p}.(\vec{w}_h-\tilde{\vec{w}}_h)\sqrt{a}\, d\xi^1 d\xi^2| \leq c\, h\, ||\vec{p}||_{0,\Omega}\, ||\vec{w}_h|| \quad .$$

Next, by virtue of theorem 5.2.5 and estimate (5.2.60), we obtain

$$|\int_\Omega \vec{p}.\tilde{\vec{w}}_h(\sqrt{a_h}-\sqrt{a})d\xi^1 d\xi^2| \leq c\, h\, ||\vec{p}||_{0,\Omega}\, ||\vec{w}_h|| \quad ,$$

so that we deduce the existence of a constant c, independent of h, such that

$$|f(\vec{w}_h) - \tilde{f}_h(\tilde{\vec{w}}_h)| \leq c\, h\, ||\vec{p}||_{0,\Omega}\, ||\vec{w}_h|| \quad . \tag{5.3.2}$$

<u>Step 3</u> : <u>Estimate of</u> $|\tilde{f}_h(\tilde{\vec{w}}_h)-f_h^*(\tilde{\vec{w}}_h)|$.

We find from (5.2.60)

$$\sqrt{a_h} = \sqrt{a} + O(h) ,$$

so that, for h sufficiently small, the function $\sqrt{a_h}$ is uniformly bounded in $\bar{\Omega}$, independently of h.

Then, as an immediate consequence of CIARLET [5, Theorem 4.1.5] we deduce from properties (5.2.143) and (5.2.144) the existence of a constant c, independent of h, such that, for h sufficiently small,

$$|\tilde{f}_h(\tilde{\vec{w}}_h) - f_h^*(\tilde{\vec{w}}_h)| \leq c\ h\ ||\vec{p}||_{1,q,\Omega}\ ||\tilde{\vec{w}}_h||_h\ ,$$

or, by using Theorem 5.2.5,

$$|\tilde{f}_h(\tilde{\vec{w}}_h) - f_h^*(\tilde{\vec{w}}_h)| \leq c\ h\ ||\vec{p}||_{1,q,\Omega}\ ||\vec{w}_h||\ . \tag{5.3.3}$$

Estimate (5.3.1) immediatly follows by combining (5.3.2) and (5.3.3). □

5.4 - <u>The bilinear form</u> $b_h(.,.)$ <u>is uniformly</u> $\vec{V}_h$<u>-elliptic</u>

In Theorem 5.1.1 which gives the abstract error estimate, we have assumed the property (5.1.1), i.e. the bilinear form $b_h(.,.)$ is uniformly $\vec{V}_h$-elliptic. In the next theorem we check that this property is effectively satisfied.

<u>Theorem 5.4.1</u> : <u>Assume that</u>

$$\forall\ \hat{\phi} \in P_0(\hat{K}) \quad , \quad \hat{E}_1(\hat{\phi}) = 0\ , \qquad \text{(for membrane terms)} \tag{5.4.1}$$

$$\forall\ \hat{\phi} \in P_2(\hat{K}) \quad , \quad \hat{E}_2(\hat{\phi}) = 0\ , \qquad \text{(for bending terms)} \tag{5.4.2}$$

<u>with</u> $\hat{E}_1$ <u>and</u> $\hat{E}_2$ <u>defined by</u> (5.2.138) <u>and</u> (5.2.141). <u>Then there exists constants</u> $\varepsilon_1 > 0$, $h_1 > 0$, $\alpha > 0$ <u>such that for any</u> $\varepsilon(0<\varepsilon<\varepsilon_1)$, <u>for any</u> $h(0<h<h_1)$, <u>for any shell</u> $\mathcal{S} \in \mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, w_\alpha)$, <u>we have</u>

$$\alpha\ ||\vec{v}_h||^2 \leq b_h(\vec{v}_h, \vec{v}_h) \quad , \quad \forall\ \vec{v}_h \in \vec{V}_h\ . \tag{5.4.3}$$

Proof : The assumptions of Theorem 5.2.8 are satisfied. Then

$$|a(\vec{v}_h,\vec{v}_h)-b_h(\vec{v}_h,\vec{v}_h)| \leq C(h+\varepsilon)\ \|v_h\|^2 \ . \tag{5.4.4}$$

Moreover from BERNADOU-CIARLET [15, Theorem 6.1.3], the bilinear form a(.,.) is V-elliptic, i.e., there exists a constant $\beta > 0$ such that

$$\beta\|\vec{v}\|^2 \leq a(\vec{v},\vec{v}) \quad , \quad \forall\vec{v} \in \vec{V} \quad .$$

Of course, this constant $\beta > 0$ is depending of the geometrical form of the middle surface. By restricting this study to the set $\mathcal{S}_o(\rho_\alpha,e_\alpha,\omega_\alpha)$ defined through the compact sets $[0,\rho_1]$, $[\rho_2,\rho_1]$, $[e_1,e_2]$, we deduce the existence of a number $\beta_o > 0$ such that

$$\beta_o\ \|\vec{v}\|^2 \leq a(\vec{v},\vec{v}) \quad , \quad \forall\vec{v} \in \vec{V} \quad , \quad \forall\mathcal{S} \in \mathcal{S}_o(\rho_\alpha,e_\alpha,\omega_\alpha) \quad . \tag{5.4.5}$$

Then, from relations (5.4.4) (5.4.5) and from the inclusion $\vec{V}_h \subset \vec{V}$, we obtain

$$\left.\begin{array}{l} b_h(\vec{v}_h,\vec{v}_h) \leq \ [\beta_o - c(h+\varepsilon)]\ \|\vec{v}_h\|^2 \\ \forall\vec{v}_h \in \vec{V}_h \quad , \quad \forall\mathcal{S} \in \mathcal{S}_\varepsilon(\rho_\alpha,e_\alpha,\omega_\alpha) \ . \end{array}\right\}$$

The estimate (5.4.3) arises by choosing h and ε sufficiently small, that is to say $0 < h < h_1 \leq h_o$ and $0 < \varepsilon < \varepsilon_1 \leq \varepsilon_o$.

□

5.5. About Carr's approach

As it was shown in section 5.2, the discontinuity between tangential and transverse displacements at the common edge of two facets involves extraneous bending effects. A natural way of reconsidering the approximation problem would consist in taking the same order of polynomials for the interpolation of the three components of the displacement (for example, one can define the discrete spaces as $\approx\!\!\!\!\approx X_h = (\tilde{\tilde{X}}_{h2})^3$).

Following the Carr's alternative presented in section 4.4, we prove here that no major improvement is to be expected in raising the order of interpolation of the tangential components of the displacement. For clarity, we will disregard the effects of numerical integration. Of course, we use again hereunder, most of the results obtained in sections 5.1 and 5.2. In this sense, we will only consider the estimate $|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{\ \alpha}_{h\beta}(\tilde{\vec{v}}_h)|_{0,K}$.

Theorem 5.5.1 : There exists constants $c^{\ell}_{j\alpha\beta}$, $c^{\ell\lambda}_{j\alpha\beta}$, $c^{k\varepsilon\eta}_{j\alpha\beta}$, independent of h, such that, for any $\tilde{\vec{v}}_h \in \tilde{\vec{X}}_h$ and $\vec{v}_h \in \vec{X}_h$ associated through the bijection F_h defined in theorem 4.4.1, we have :

$$\tilde{\rho}_{h\alpha\beta}(\tilde{\vec{v}}_h)\Big|_{K_j} = \bar{\rho}_{\alpha\beta}(\vec{v}_h) + a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi) \times$$
$$\times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\alpha\beta}] \qquad (5.5.1)$$
$$+ O(h) \sum_{k=1}^{3} [c^{\ell}_{j\alpha\beta}v_{h\ell}(\bar{\xi}_k)+c^{\ell\lambda}_{j\alpha\beta}v_{h\ell,\lambda}(\bar{\xi}_k)+c^{k\varepsilon\eta}_{j\alpha\beta}v_{hk,\varepsilon\eta}(\bar{\xi}_k)]$$

where the expression of $A^{\varepsilon\mu}_{\lambda}$ is given by relation (5.2.26).

Proof : From the definitions (5.2.35) and (5.2.36), we have :

$$\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h) = a^{\alpha\nu}\{v_{h3|\nu\beta} + b^{\lambda}_{\beta|\nu}v_{h\lambda} + b^{\lambda}_{\beta}v_{h\lambda|\nu} + b^{\lambda}_{\nu}v_{h\lambda|\beta} - c_{\nu\beta}v_{h3}\} \qquad (5.5.2)$$

$$\tilde{\rho}^{\ \alpha}_{h\beta}(\tilde{\vec{v}}_h) = a^{\alpha\nu}_h\tilde{v}_{h3,\nu\beta} , \qquad (5.5.3)$$

and by definition of the space $\tilde{X}_{h2}$, we have on any subtriangle K_j, j=1,2,3, of a given triangle K with vertices $\Sigma_i = (\xi^1_i,\xi^2_i)$, i=1,2,3 :

$$\tilde{v}_{h3,\alpha\beta}(\xi)\Big|_{K_j} = \sum_{i=1}^{3} (p^o_{j,i}(\lambda))_{,\alpha\beta}\tilde{v}_{h3}(\Sigma_i) +$$
$$\sum_{i=1}^{3} \{(\xi^{\nu}_{i+1}-\xi^{\nu}_{i})(p^1_{j,i,i+1}(\lambda))_{,\alpha\beta}+(\xi^{\nu}_{i-1}-\xi^{\nu}_{i})(p^1_{j,i,i-1}(\lambda))_{,\alpha\beta}\}\tilde{v}_{h3,\nu}(\Sigma_i) \qquad (5.5.4)$$

By virtue of the compatibility relations (4.4.2), we get with (5.2.17) :

$$\tilde{v}_{h3}(\Sigma_i) = d^j_{h3}(\Sigma_i)v_{hj}(\Sigma_i) \quad , \quad i = 1, 2, 3, \qquad (5.5.5)$$

and, from (5.2.62), we derive the finite expansion (5.2.64), i.e., for i=1,2,3 :

$$\begin{aligned}\tilde{v}_{h3}(\Sigma_i) &= v_{h3}(\Sigma_i) + \{[1-R^{\lambda}_{\lambda 1}(\xi)]D^{\nu}_{31}(\xi,\Sigma_i) + D^{\nu}_{32}(\xi,\Sigma_i)\}\ v_{h\nu}(\Sigma_i)\\ &+ \frac{1}{2}\{2D^{3}_{32}(\xi,\Sigma_i) + (R^{\lambda}_{\lambda 1}(\xi))^2 - R(\xi)\}\ v_{h3}(\Sigma_i) + O(h^3)c^{j}_{3i}v_{hj}(\Sigma_i)\end{aligned} \tag{5.5.6}$$

Otherwise, from (4.4.19), (4.4.20), (4.4.24), we have directly :

$$\left.\begin{aligned}\tilde{v}_{h3,\nu}(\Sigma_i) &= e_{\mu\nu}e^{\kappa\lambda}\sqrt{\frac{a_h}{a(\Sigma_i)}}\ \{(\vec{a}^{\mu}_{h}.\vec{a}_{\kappa}(\Sigma_i))[v_{h3,\lambda}(\Sigma_i)+b^{\varepsilon}_{\lambda}(\Sigma_i)v_{h\varepsilon}(\Sigma_i)]\\ &+ \frac{1}{2}\ (\vec{a}^{\mu}_{h}.\vec{a}_{3}(\Sigma_i))v_{h\lambda,\kappa}(\Sigma_i)\}\ ,\end{aligned}\right\} \tag{5.5.7}$$

Subsequently, we are going to prove that this expression (5.5.7) is nothing else than relation (5.2.45) under an $O(h^2)$-estimate. Indeed, rewrite the compatibility relations (4.4.3) in the following way, by using definitions (5.2.17) :

$$\frac{1}{\sqrt{a_h}}\ e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)d^{\lambda}_{h\mu}(\Sigma_i) = B^{\lambda}(\Sigma_i) - d^{\lambda}_{h3}(\Sigma_i)\tilde{\omega}^{3}_{h}(\Sigma_i)\ , \tag{5.5.8}$$

$$\tilde{\omega}^{3}_{h}(\Sigma_i)d^{3}_{h3}(\Sigma_i) = B^{3}(\Sigma_i) - \frac{1}{\sqrt{a_h}}\ e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)d^{3}_{h\mu}(\Sigma_i)\ . \tag{5.5.9}$$

Now by substituting (5.5.9) into (5.5.8), we get :

$$\left.\begin{aligned}&e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)\ [d^{\lambda}_{h\mu}(\Sigma_i) - \frac{d^{\lambda}_{h3}(\Sigma_i)d^{3}_{h\mu}(\Sigma_i)}{d^{3}_{h3}(\Sigma_i)}\ ] =\\ &\sqrt{a_h}\ \{B^{\lambda}(\Sigma_i) - \frac{d^{\lambda}_{h3}(\Sigma_i)}{d^{3}_{h3}(\Sigma_i)}\ B^{3}(\Sigma_i)\}\ .\end{aligned}\right\} \tag{5.5.10}$$

By noticing that we have, from (5.2.51) (5.2.52) and (5.2.62) :

$$\frac{d^{\lambda}_{h3}(\Sigma_i)d^{3}_{h\mu}(\Sigma_i)}{d^{3}_{h3}(\Sigma_i)} = O(h^2)\ ,$$

and, in (5.5.9) :

$$\frac{d^{\lambda}_{h3}(\Sigma_i)}{d^{3}_{h3}(\Sigma_i)}\ B^{3}(\Sigma_i) = d^{\lambda}_{h3}(\Sigma_i)\tilde{\omega}^{3}_{h}(\Sigma_i) + O(h^2)\{c^{\nu}\tilde{v}_{h3,\nu}(\Sigma_i)\}\ ,$$

we derive in (5.5.10) :

$$\left.\begin{array}{l} e^{\mu\nu}\tilde{v}_{h3,\nu}(\Sigma_i)d_{h\mu}^{\lambda}(\Sigma_i) = \\ \qquad = \sqrt{a_h}\ \{B^{\lambda}(\Sigma_i)-d_{h3}^{\lambda}(\Sigma_i)\tilde{\omega}_h^3(\Sigma_i)\} + 0(h^2)\{\tilde{c}^{\nu}\tilde{v}_{h3,\nu}(\Sigma_i)\} \end{array}\right\} \quad (5.5.11)$$

Therefore, under an $0(h^2)$-estimate, the values $\tilde{v}_{h3,\nu}(\Sigma_i)$ are solutions of the same system as the one obtained in step 2 of the proof of Theorem 5.2.2, and we deduce that the expression (5.5.7) is an $0(h^2)$-approximation of the previous expression (5.2.45) obtained for the Clough-Johnson method.

So, by using the technical developments of the steps 3 to 7 of Theorem 5.2.2, we obtain the estimate (5.5.1).

□

Consequently, conclusions about this approach follows exactly in the same manner as in the former study.

6 - PSEUDO-CONVERGENCE AND ERROR ESTIMATES ; A NEW CONVERGENT METHOD

6.1 - Pseudo-convergence and error estimates for shallow shells

Now we can prove the error estimate theorem.

Theorem 6.1.1 : Let $\mathcal{C}_h$ be a regular family of triangulations of the domain Ω. Let $\vec{V}_h$, $\vec{\vec{V}}_h$ be the associated finite element spaces respectively defined by (3.1.4) (3.1.5) and (4.2.15) (4.2.38).

Assume that the numerical integration schemes (5.2.138) and (5.2.141) satisfy the followin conditions

$$\forall\hat{\phi} \in P_o(\hat{K}) \quad , \quad \hat{E}_1(\hat{\phi}) = 0 \quad , \text{(for membrane terms)} \qquad (6.1.1)$$

$$\forall\hat{\phi} \in P_2(\hat{K}) \quad , \quad \hat{E}_2(\hat{\phi}) = 0 \quad , \text{(for bending terms)} \qquad (6.1.2)$$

<u>If the solution</u> $\vec{u} \in \vec{V}$ <u>of the variational problem</u> (2.2.5) <u>belongs to the space</u> $H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$, <u>if the loads</u> $\vec{p}$ <u>belong to the space</u> $(W^{1,q}(\Omega))^3$, $q \in \mathbb{R}$, $q > 2$, <u>then there exists constants</u> $\varepsilon_1 > 0$, $h_1 > 0$, $c > 0$ <u>such that for any</u> $\varepsilon(0<\varepsilon<\varepsilon_1)$, <u>for any</u> $h(0<h<h_1)$, <u>for any shell</u> $\mathscr{S} \in \mathscr{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$, <u>we have</u>

$$\|\vec{u}-\vec{u}_h^*\| \leq c\ [h\{(\sum_{\alpha=1}^{2} \|u_\alpha\|^2_{2,\Omega} + \|u_3\|^2_{3,\Omega})^{1/2} + \|\vec{p}\|_{1,q,\Omega}\} + \varepsilon\|\vec{u}\|] \tag{6.1.3}$$

<u>where</u> $\vec{u}_h^* \in \vec{V}_h$ <u>is the solution of the discret problem</u> (4.3.19).

<u>Proof</u> : (i) The properties of interpolation of the triangles of type (1) and of the reduced H.C.T.-triangle involve (CIARLET [5])

$$\|u_\alpha - \pi_{h1} u_\alpha\|_{1,\Omega} \leq c\ h |u_\alpha|_{2,\Omega}$$

$$\|u_3 - \pi_{h2} u_3\|_{2,\Omega} \leq c\ h \|u_3\|_{3,\Omega}$$

where π_{h1} , π_{h2} denote the corresponding interpolation operator. Thus we have

$$\|\vec{u}-\pi_h\vec{u}\| \leq c\ h\ (\sum_{\alpha=1}^{2} \|u_\alpha\|^2_{2,\Omega} + \|u_3\|^2_{3,\Omega})^{1/2} \quad , \tag{6.1.4}$$

where $\pi_h\vec{u} = (\pi_{h1}u_1, \pi_{h1}u_2, \pi_{h2}u_3)$.

(ii) From Theorem 5.2.8 we have

$$|a(\pi_h\vec{u},\vec{w}_h) - b_h(\pi_h\vec{u},\vec{w}_h)| \leq c(h+\varepsilon)\ \|\pi_h\vec{u}\|\ \|\vec{w}_h\|$$

so that

$$\sup_{\vec{w}_h \in \vec{V}_h} \frac{|a(\pi_h\vec{u},\vec{w}_h) - b_h(\pi_h\vec{u},\vec{w}_h)|}{\|\vec{w}_h\|} \leq c(h+\varepsilon)\ \|\pi_h\vec{u}\| \tag{6.1.5}$$

(iii) From Theorem 5.3.1 we obtain

$$\sup_{\vec{w}_h \in \vec{V}_h} \frac{|f(\vec{w}_h) - g_h(\vec{w}_h)|}{\|\vec{w}_h\|} \leq c\ h\ \|\vec{p}\|_{1,q,\Omega} \quad . \tag{6.1.6}$$

(iv) Let us record that the relation (5.1.1) is satisfied (Theorem 5.4.1). Thus, we can apply Theorem 5.1.1. The infinimum is bounded from above taking $\vec{v}_h = \pi_h \vec{u}$. Since $\|\pi_h \vec{u}\| \leq c \, \|\vec{u}\|$, we obtain (6.1.3) by using (6.1.4) (6.1.5) and (6.1.6).

□

Remark 6.1.1 : We have entitled this section "pseudo-convergence" since two small parameters h and ε appear in the error estimate (6.1.3) ; the first one, h, can tend asymptotically to 0, while the second one, ε, is fixed for any given "quasi-shallow" shell. Nevertherless, in shallow shell theories, such geometrical counterparts are generally considered as being "neglectible".

□

6.2 - Examples of suitable numerical integration schemes

We need two different numerical integration schemes in order to satisfy the properties (6.1.1) and (6.1.2).

The first one is immediate : it suffices to evaluate the functions to integrate at any point of the triangle, for example at the barycenter.

The second one is classical : according to STROUD [27] we shall use the following scheme :

$$\int_K \phi(x)\,dx \sim \frac{1}{3} \sum_{i=1}^{3} \phi(b_i)$$

where b_i denote the midside points of the triangle K.

6.3 - A new convergent method for general shells

In the statement of Theorems 5.2.3, 5.2.4, 5.2.8, 5.4.1 and 6.1.1, we have introduced a restriction upon the geometry of the middle surface of the shell. This restriction was motivated by the treatment of the residual term in the estimate (5.2.37). In this paragraph, we derive a new approximation method which is unconditionnally convergent, whatever the geometry of the shell may be.

Let us record the relation (5.2.37) : there exists constants $c^{\alpha\ell}_{\beta j}$, $c^{\alpha\ell\lambda}_{\beta j}$, $c^{\alpha\varepsilon\eta}_{\beta j}$, independent of h, such that :

$$\left.\begin{aligned}
\tilde{\rho}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h)\Big|_{K_j} &= \bar{\rho}^{\alpha}_{\beta}(\vec{v}_h) + \\
&+ a^{\alpha\nu}_h(\xi)\gamma^{\lambda}_{\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi) \times \\
&\times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\nu\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\nu\beta}] \\
&+ O(h) \sum_{k=1}^{3} [c^{\alpha\ell}_{\beta j}v_{h\ell}(\bar{\xi}_k)+c^{\alpha\ell\lambda}_{\beta j}v_{h\ell,\lambda}(\bar{\xi}_k)+c^{\alpha\varepsilon\eta}_{\beta j}v_{h3,\varepsilon\eta}(\bar{\xi}_k)] \ ,
\end{aligned}\right\} \quad (6.3.1)$$

(relation 6.3.1 is easily obtained from (5.2.36) (5.2.37) and $a^{\alpha\nu}_h = a^{\alpha\nu}(\xi) + O(h)$). As far as the second fundamental form ($b_{\alpha\beta}$) may be evaluated - at least at the numerical integration nodes -, we propose here another way to take into account the residual term in (6.3.1).

According to (5.2.30), we have

$$\tilde{\gamma}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h) = \gamma^{\alpha}_{\beta}(\vec{v}_h) + O(h)\ \{c^{\alpha j}_{\beta}v_{hj}(\xi)+c^{\alpha\nu\varepsilon}_{\beta}v_{h\nu,\varepsilon}+\sum_{i=1}^{3} c^{\alpha\varepsilon}_{\beta i}v_{h3,\varepsilon}(\bar{\xi}_i)\} \ , \quad (6.3.2)$$

so that relation (6.3.1) can be equivalently written as :

$$\left.\begin{aligned}
\tilde{\rho}^{\alpha}_{h\beta}(\tilde{\vec{v}}_h)\Big|_{K_j} &= \bar{\rho}^{\alpha}_{\beta}(\vec{v}_h) + \\
&+ a^{\alpha\nu}_h\ \tilde{\gamma}^{\lambda}_{h\eta}(\tilde{\vec{v}}_h)b_{\varepsilon\mu}(\xi) \times \\
&\times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_{i})(p^1_{j,i,i+1})_{,\nu\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_{i})(p^1_{j,i,i-1})_{,\nu\beta}] \\
&+ O(h) \sum_{k=1}^{3} \{\tilde{c}^{\alpha\ell}_{\beta j}v_{h\ell}(\bar{\xi}_k)+\tilde{c}^{\alpha\ell\lambda}_{\beta j}v_{h\ell,\lambda}(\bar{\xi}_k)+c^{\alpha\varepsilon\eta}_{\beta j}v_{h3,\varepsilon\eta}(\bar{\xi}_k)\}
\end{aligned}\right\} \quad (6.3.3)$$

Then, we introduce a "perturbed" expression $\tilde{\rho}^{*}_{h}$ of the change of curvature $\tilde{\rho}$, defined as follows :

$$\left.\begin{aligned}\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\tilde{v}}_h)\big|_{K_j} &= \tilde{\rho}^{\ \alpha}_{h\beta}(\vec{\tilde{v}}_h)\big|_{K_j} + \\ &- b_{\varepsilon\mu}(\xi)a_h^{\alpha\nu}\ \tilde{\gamma}^{\ \lambda}_{h\eta}(\vec{\tilde{v}}_h) \times \\ &\qquad \times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\nu\beta}+(\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\nu\beta}]\end{aligned}\right\}\quad(6.3.4)$$

where $A^{\varepsilon\mu}_{\lambda}(\Sigma_i)$ is given by (5.2.97). Following (5.2.36) and (5.2.38), one gets the explicit formulation :

$$\left.\begin{aligned}\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\tilde{v}}_h)\big|_{K_j} &= a_h^{\alpha\nu}\ \{\sum_{i=1}^{3}\ (p^o_{j,i})_{,\nu\beta}\tilde{v}_{h3}(\Sigma_i) + \\ &+ \sum_{i=1}^{3}\ [(\xi^\eta_{i+1}-\xi^\eta_i)(p^1_{j,i,i+1})_{,\nu\beta}+(\xi^\eta_{i-1}-\xi^\eta_i)(p^1_{j,i,i-1})_{,\nu\beta}] \times \\ &\qquad \times [\tilde{v}_{h3,\eta}(\Sigma_i)-A^{\varepsilon\mu}_{\lambda}(\Sigma_i)b_{\varepsilon\mu}(\xi)\tilde{\gamma}^{\ \lambda}_{h\eta}(\vec{\tilde{v}}_h)]\ ,\end{aligned}\right\}\quad(6.3.5)$$

where it is infered from (5.2.12) that $\tilde{\gamma}^{\ \lambda}_{h\eta}$ is constant over each triangle K.

The new discrete problem can be stated as (see (4.3.14) (4.3.18)) :

<u>Problem 6.3.1</u> : <u>Find</u> $\vec{\tilde{u}}^*_h \in \vec{\tilde{V}}_h$ <u>such that</u>

$$\tilde{a}^{**}_h(\vec{\tilde{u}}^{**}_h, \vec{\tilde{v}}_h) = f^*_h(\vec{\tilde{v}}_h) \quad , \quad \forall \vec{\tilde{v}}_h \in \vec{\tilde{V}}_h \qquad (6.3.6)$$

<u>with</u>

$$\begin{aligned}\tilde{a}^{**}_h(\vec{\tilde{u}}_h, \vec{\tilde{v}}_h) = &\sum_{K\in\mathscr{T}_h} \sqrt{a_h}\ [\sum_{\ell_1=1}^{L_1} \omega_{\ell 1,K}\ (\frac{Ee}{1-\nu^2}\ \{(1-\nu)\tilde{\gamma}^{\ \alpha}_{h\beta}(\vec{\tilde{u}}_h)\tilde{\gamma}^{\ \beta}_{h\alpha}(\vec{\tilde{v}}_h) + \\ &+ \nu\tilde{\gamma}^{\ \alpha}_{h\alpha}(\vec{\tilde{u}}_h)\tilde{\gamma}^{\ \beta}_{h\beta}(\vec{\tilde{v}}_h)\})\ (b_{\ell_1,K}) \\ &+ \sum_{i=1}^{3}\ \sum_{\ell_2=1}^{L_2} \omega_{\ell_2,K_i}\ (\frac{Ee^3}{12(1-\nu^2)}\ \{(1-\nu)\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\tilde{u}}_h)\tilde{\rho}^{*\beta}_{h\alpha}(\vec{\tilde{v}}_h) + \\ &+ \nu\tilde{\rho}^{*\alpha}_{h\alpha}(\vec{\tilde{u}}_h)\tilde{\rho}^{*\beta}_{h\beta}(\vec{\tilde{v}}_h)\})\ (b_{\ell_2,K_i})]\ ,\end{aligned} \qquad (6.3.7)$$

where the tensors $\tilde{\gamma}^{\ \alpha}_{h\beta}$ and $\tilde{\rho}^{*\alpha}_{h\beta}$ are defined in (4.3.1) and (6.3.4), and where the linear form $f^*_h(.)$ is defined in (4.3.15).

Now, by using the bijection F_h defined in Theorem 4.2.1, the problem 6.3.1 is equivalent to :

<u>Find</u> $\vec{u}_h^{\star} \in \vec{V}_h$ <u>such that</u>

$$b_h^{\star}(\vec{u}_h^{\star\star}, \vec{v}_h) = g_h(\vec{v}_h) \quad , \quad \forall \vec{v}_h \in \vec{V}_h \quad , \tag{6.3.8}$$

where, by definition, for any $\vec{v}_h, \vec{w}_h \in \vec{V}_h$ in correspondence with $\vec{\tilde{v}}_h, \vec{\tilde{w}}_h \in \vec{\tilde{V}}_h$ through the bijection F_h, we set :

$$b_h^{\star}(\vec{v}_h, \vec{w}_h) = \tilde{a}_h^{\star\star}(\vec{\tilde{v}}_h, \vec{\tilde{w}}_h) \quad . \tag{6.3.9}$$

Consequently, we can establish the convergence of this new method for *general* shell problems :

<u>Theorem 6.3.1</u> : <u>Let</u> $\mathcal{C}_h$ <u>be a regular family of triangulations of the domain</u> Ω. <u>Let</u> $\vec{V}_h, \vec{\tilde{V}}_h$ <u>be the associated finite element spaces respectively defined by</u> (3.1.4) (3.1.5) <u>and</u> (4.2.15) (4.2.38).

<u>Assume that the numerical integration schemes</u> (5.2.138) <u>and</u> (5.2.141) <u>satisfy the following conditions</u>

$$\forall \hat{\phi} \in P_o(\hat{K}) \quad , \quad \hat{E}_1(\hat{\phi}) = 0 \quad , \quad \text{(for membrane terms)} \tag{6.3.10}$$

$$\forall \hat{\phi} \in P_2(\hat{K}) \quad , \quad \hat{E}_2(\hat{\phi}) = 0 \quad , \quad \text{(for bending terms)} \quad . \tag{6.3.11}$$

<u>Then, if the solution</u> $\vec{u} \in \vec{V}$ <u>of the variational problem</u> (2.2.5) <u>belongs to the space</u> $H^2(\Omega) \times H^2(\Omega) \times H^3(\Omega)$, <u>if the loads</u> $\vec{p}$ <u>belongs to the space</u> $(W^{1,q}(\Omega))^3$, $q \in \mathbb{R}$, $q > 2$, <u>there exists a constant</u> c, <u>independent of</u> h, <u>and</u> $h_1 > 0$ <u>such that, for any</u> $h < h_1$, <u>we have</u>

$$\|\vec{u} - \vec{u}_h^{\star\star}\| \le ch\{(\sum_{\alpha=1}^{2} \|u_\alpha\|_{2,\Omega}^2 + \|u_3\|_{3,\Omega}^2)^{1/2} + \|\vec{p}\|_{1,q,\Omega}\} \tag{6.3.12}$$

<u>where</u> $\vec{u}_h^{\star\star} \in \vec{V}_h$ <u>is the solution of the discrete problem</u> (6.3.8).

<u>Proof</u> : The proof follows exactly the lines of paragraphs 5 and 6.1. The major difference comes from Theorem 5.2.3 where the estimate (5.2.118) is replaced by

$$|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{*\alpha}_{h\beta}(\vec{\tilde{v}}_h)|_{0,K} \le ch\{\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{2,K}\}^{1/2} . \qquad (6.3.13)$$

Thus, this estimate is completely independent of the magnitude of the parameter ε, and the proof is pursued without taking into account any restriction upon the class of the shell : results of Theorems 5.2.8 and 5.4.1 are now completely independent of the magnitude of the parameter ε.

□

<u>Remark 6.3.1</u> : It is clear that this study has been performed under the assumption that an explicit knowledge of the geometry of the shell is available. For engineering computations, it should be interesting to define an approximation method of thin shell problems by flat plate elements which only relies on the euclidean coordinates of the vertices of the facets. We will propose such a method in a further report.

□

<u>Remark 6.3.2</u> : Thus, this "perturbation" method assures the convergence of the "facet approximation" to the solution of the continuous problem corresponding to the use of $\bar{\rho}_{\alpha\beta}$ as a tensor of change or curvature. To obtain the same type of result when using tensor $\rho_{\alpha\beta}$, defined in (2.1.15), instead of $\bar{\rho}_{\alpha\beta}$, it suffices to modify the perturbation (6.3.4). Indeed, from relation (5.2.91), use

$$\begin{aligned}\tilde{\rho}^{**\alpha}_{h\ \beta}(\vec{\tilde{v}}_h)\big|_{K_j} = \tilde{\rho}_{h\beta}^{\ \alpha}(\vec{\tilde{v}}_h)\big|_{K_j} &- \frac{1}{2}\{b^{\alpha}_{\nu}(\xi)\tilde{\gamma}_{h\beta}^{\ \nu}(\vec{\tilde{v}}_h) + b^{\nu}_{\beta}(\xi)\tilde{\gamma}_{h\nu}^{\ \alpha}(\vec{\tilde{v}}_h)\} \\ &- b_{\varepsilon\mu}(\xi)a^{\alpha\nu}_h\tilde{\gamma}_{h\eta}^{\ \lambda}(\vec{\tilde{v}}_h) \times \\ &\quad \times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_i)(p^1_{j,i,i+1})_{,\nu\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_i)(p^1_{j,i,i-1})_{,\nu\beta}]\end{aligned}$$

instead of the expression (6.3.4) ; then, the estimate (6.3.13) becomes now :

$$|\rho^{\alpha}_{\beta}(\vec{v}_h)-\tilde{\rho}^{**\alpha}_{h\ \beta}(\vec{\tilde{v}}_h)|_{0,K} \leq ch\{\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{2,K}\}^{1/2} \ . \qquad \square$$

CONCLUSION

Without any doubt, the *main result* of this study is the estimate (5.2.37) stated in Theorem 5.2.2. From this estimate, we can draw several lessons :

i) the approximation of thin shell problems by the CLOUGH-JOHNSON flat plate elements is not always convergent when applied to general shells. The convergence is directly depending of the magnitude of the second term of the right hand member of estimate (5.2.37), i.e., with notation (5.2.101) :

$$c_{\alpha\beta} \stackrel{\text{def}}{=} a^{\nu\lambda}(\xi)\gamma_{\nu\eta}(\vec{v}_h)b_{\varepsilon\mu}(\xi) \times$$
$$\times \sum_{i=1}^{3} A^{\varepsilon\mu}_{\lambda}(\Sigma_i)[(\xi^{\eta}_{i+1}-\xi^{\eta}_i)(p^1_{j,i,i+1})_{,\alpha\beta}+(\xi^{\eta}_{i-1}-\xi^{\eta}_i)(p^1_{j,i,i-1})_{,\alpha\beta}]$$

ii) in this study, we have proved that :

* for quasi-shallow shells (see Theorem 5.2.3) :

$$|c_{\alpha\beta}|_{0,K} \leq c_1\varepsilon\,\|\vec{v}_h\|$$

* for a "non-shallow" circular cylinder, this term cannot be neglected (see Lemma 5.2.1). This result corroborates several numerical observations by ZIENKIEWICZ [23, p. 239], IRONS and AHMAD [24, p. 281] and GALLAGHER [25, p. 5].

iii) the field of application of such a method can be probably extended to some classes of non-shallow shells by energy considerations. For instance, we could restrict our attention to the case of preponderant flexural strains and develop the ideas of Remark 5.2.3.

iv) finally, it is possible to derive a new method which is convergent for any general shells without any restrictions upon the geometry. This important improvement is obtained by means of a slight perturbation of the bending term.

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